Homework Set 2 Solutions

1. (4 points) Let W be a finite subset of $\mathcal{P}$, the vector space of all polynomials. Furthermore, let no two elements of W have the same degree. Show that W is linearly independent.

Solution. Let $W = \{p_j(x)\}_1^n$, and let d_j be the degree of p_j. Furthermore, order the elements such that $d_1 \le d_2 \le \cdots \le d_n$. To check if the set is linearly independent, we examine

$$\sum_{j=1}^{n} c_j p_j = 0.$$

Matching powers of x, we have

$$c_n a_{nd_n} x^{d_n} + \text{lower order terms} = 0.$$

Therefore, we see that $c_n = 0$, since $a_{nd_n} \neq 0$ or d_n would not be the degree. But then the sum to check is

$$\sum_{j=1}^{n-1} c_j p_j = 0,$$

which is of the same form with n replaced by $n-1$. Therefore, working backwards, we see that $c_j = 0$ for all j and hence W is linearly independent.

2. (2 points per part) Determine whether each of the following sets is linearly independent:

(a)

$$\left\{ \begin{pmatrix} -1 & 2 \\ 1 & -5 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ -2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 6 \\ 0 & 3 \end{pmatrix} \right\}$$

Solution. We use the linear combination test. Using the zero entries in the second and third matrices, we see that to match up the entries in the first column, we must have that

$$\begin{pmatrix} -1 & 2 \\ 1 & -5 \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} 0 & 2 \\ -2 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 6 \\ 0 & 3 \end{pmatrix}.$$

But then the entries in the second column do not match. Thus the set is linearly independent, since trivially the second and third matrices are not multiples of each other.

(b) $\{\sin^3 x,\ \sin x,\ \sin 3x\}$

Solution. Using trigonometric identities, we have that

$$\sin^3 x = \frac{(\sin x)(1-\cos 2x)}{2} = \frac{\sin x}{2} - \frac{\sin 3x + \sin(-x)}{4} = \frac{3\sin x - \sin 3x}{4}.$$

Thus the set is linearly dependent.

(c) $\{(1,3,2),(2,-1,4),(-4,9,-8)\}$

Solution. If the set is linearly independent, the following equations have a nontrivial solution:

$$\begin{aligned} c_1 + 2c_2 - 4c_3 &= 0 \\ 3c_1 - c_2 + 9c_3 &= 0 \\ 2c_1 + 4c_2 - 8c_3 &= 0 \end{aligned}$$

But the third equation is a multiple of the first, so the equations are redundant and the set is linearly dependent.

3. (5 points) Show that a finite set of vectors S is linearly independent if and only if every subset of S is linearly independent.

Solution. Assume that $S = \{\mathbf{s}_i\}_1^n$ is linearly independent. Let T be a subset of S. By the definition of linear independence of S, we have that

$$\sum_{i=1}^{n} c_i\mathbf{s}_i = \sum_{\mathbf{s}_i \in T} c_i\mathbf{s}_i + \sum_{\mathbf{s}_i \notin T} c_i\mathbf{s}_i = \mathbf{0}$$

implies that $c_i = 0$ for every i. Using this fact in the second sum above, we have

$$\sum_{\mathbf{s}_i \in T} c_i\mathbf{s}_i = \mathbf{0} \quad \Longrightarrow \quad c_i = 0\ \forall i.$$

Since T is arbitrary, one direction is proved. Now assume that every proper subset of S is linearly independent, and assume that S is linearly dependent. Then we have that

$$\sum_{i=1}^{n} c_i\mathbf{s}_i = \mathbf{0}$$

with not all the $c_i = 0$. Let T be the set of vectors $\mathbf{s}_i$ such that $c_i \neq 0$. But then we have

$$\sum_{\mathbf{s}_i \in T} c_i\mathbf{s}_i = \mathbf{0}, \qquad c_i \neq 0\ \forall i,$$

and hence T is linearly dependent, a contradiction.

4. (3 points) Let $S_1,\ S_2$ be two subsets of a vector space V. Prove that $\text{Span}(S_1 \cup S_2) = \text{Span}(S_1) +\ \text{Span}(S_2)$.

Solution. Solution. Let $S_1 = \{\mathbf{v}_i\}_1^n, S_2 = \{\mathbf{v}_i\}_p^m$, $1 \le p \le n \le m$, $\mathbf{w} \in \text{Span}(S_1 \cup S_2)$. Then

$$\mathbf{w} = \sum_{i=1}^{m} c_i\mathbf{v}_i = \sum_{i=1}^{n} d_i\mathbf{v}_i + \sum_{i=p}^{n}(c_i - d_i)\mathbf{v}_i + \sum_{i=n}^{m} c_i\mathbf{v}_i = \sum_{i=1}^{n} d_i\mathbf{v}_i + \sum_{i=p}^{m} \alpha_i\mathbf{v}_i,$$

where $\alpha_i = (c_i - d_i)$ for $p \le i \le n$, and c_i otherwise. Hence $\mathbf{w} \in \text{Span}(S_1) + \text{Span}(S_2)$ and $\text{Span}(S_1 \cup S_2) \subseteq \text{Span}(S_1)+ \text{Span}(S_2)$. Now assume that $\mathbf{u} \in \text{Span}(S_1)+ \text{Span}(S_2)$. So

$$\mathbf{w} = \sum_{i=1}^{n} c_i \mathbf{v}_i + \sum_{i=p}^{m} d_i \mathbf{v}_i = \sum_{i=1}^{m} \alpha_i \mathbf{v}_i,$$

where $\alpha_i = c_i + d_i$ for $p \le i \le n$, and c_i otherwise. Hence $\mathbf{u} \in \text{Span}(S_1 \cup S_2)$ and $\text{Span}(S_1) + \text{Span}(S_2) \subseteq \text{Span}(S_1 \cup S_2)$. Hence $\text{Span}(S_1 \cup S_2) = \text{Span}(S_1)+ \text{Span}(S_2)$.

5. Let S_1, S_2 be two subsets of a vector space V.

(a) (3 points) Prove that $\text{Span}(S_1 \cap S_2) \subseteq \text{Span}(S_1) \cap \text{Span}(S_2)$.

Solution. Let $S_1 = \{\mathbf{v}_i\}_1^n, S_2 = \{\mathbf{v}_i\}_p^m$, $1 \le p \le n \le m$. Let $\mathbf{x} \in \text{Span}(S_1 \cap S_2)$. Then

$$\mathbf{x} \in \text{Span}\{\mathbf{v}_i\}_p^n$$
$$\mathbf{x} = \sum_{i=p}^{n} c_i \mathbf{v}_i = \left(\sum_{i=1}^{p-1} 0\mathbf{v}_i + \sum_{i=p}^{n} c_i \mathbf{v}_i \right) = \left[\sum_{i=p}^{n} c_i \mathbf{v}_i + \sum_{i=n+1}^{m} 0\mathbf{v}_i \right].$$

But the parenthetical expression is in $\text{Span}(S_1)$, and the bracketed expression is in $\text{Span}(S_2)$, so $\mathbf{x} \in \text{Span}(S_1) \cap \text{Span}(S_2)$. Therefore $\text{Span}(S_1 \cap S_2) \subseteq \text{Span}(S_1) \cap \text{Span}(S_2)$.

(b) (4 points) Give an example where $\text{Span}(S_1 \cap S_2) = \text{Span}(S_1) \cap \text{Span}(S_2)$ and one where $\text{Span}(S_1 \cap S_2) \subset \text{Span}(S_1) \cap \text{Span}(S_2)$.

Solution. Let $S_1 = S_2$. Then in that case we have

$$\text{Span}(S_1 \cap S_2) = \text{Span}(S_1) = \text{Span}(S_2) = \text{Span}(S_1) \cap \text{Span}(S_2).$$

Now let $S_1 = \{\mathbf{x}, \mathbf{y}\}$, $S_2 = \{\mathbf{y}, \mathbf{x} + \mathbf{y}\}$, where $\mathbf{y} \ne \alpha \mathbf{x}$. Then

$$\text{Span}(S_1 \cap S_2) = \text{Span}\,\mathbf{y},$$
$$\text{Span}(S_1) = \text{Span}\{\mathbf{x}, \mathbf{y}\} = \text{Span}\{\mathbf{x}, \mathbf{y}\} = \text{Span}(S_2) = \text{Span}(S_1) \cap \text{Span}(S_2)$$
$$\text{Span}(S_1 \cap S_2) \subset \text{Span}(S_1) \cap \text{Span}(S_2).$$

6. (2 points per part) Given a particular $A \in \mathcal{R}^{n \times n}$, and let

$$W = \{\mathbf{b} \in \mathcal{R}^n | A\mathbf{x} = \mathbf{b} \text{ for some } \mathbf{x} \in \mathcal{R}^n\}.$$

(a) Show that W is a subspace of $\mathcal{R}^n$.

Solution. Let $\mathbf{b}_1, \mathbf{b}_2 \in W$ so $A\mathbf{x}_1 = \mathbf{b}_1$ and $A\mathbf{x}_2 = \mathbf{b}_2$. Then checking the properties, we see that

$$\mathbf{b}_1 + \mathbf{b}_2 = A\mathbf{x}_1 + A\mathbf{x}_2 = A(\mathbf{x}_1 + \mathbf{x}_2) \implies \mathbf{b}_1 + \mathbf{b}_2 \in W, \quad \text{(A1)}$$
$$c\mathbf{b}_1 = cA\mathbf{x}_1 = A(c\mathbf{x}_1) \implies c\mathbf{b}_1 \in W, \quad \text{(M1)}$$
$$\mathbf{0} = A\mathbf{0} \implies \mathbf{0} \in W. \quad \text{(A3)}$$

Therefore W a subspace of $\mathcal{R}^{n\times n}$.

(b) Find a set of vectors that spans W.

Solution. Another way of writing W is

$$W = \{A\mathbf{x}\,|\,\mathbf{x}\in\mathcal{R}^n\} = \left\{\sum_{j=1}^{n} x_j\mathbf{a}_j\right\},$$

where using the definition of matrix-vector multiplication, $\mathbf{a}_j$ is the jth column of A. But in this consideration the x_j are arbitrary constants, so

$$W = \mathrm{Span}\{\mathbf{a}_j\}_1^n.$$

7. (2 points per part) Determine which of the following are bases for the listed vector spaces:

(a)

$$\left\{\begin{pmatrix} 0 & -1 & 2\\ 1 & 0 & 3\\ -2 & -3 & 0\end{pmatrix}, \begin{pmatrix} 0 & 3 & -1\\ -3 & 0 & 2\\ 1 & -2 & 0\end{pmatrix}, \begin{pmatrix} 0 & 2 & 1\\ -2 & 0 & 1\\ -1 & -1 & 0\end{pmatrix}\right\}, \quad M_- \in \mathcal{R}^{3\times 3}$$

Solution. The diagonal entries of any antisymmetric matrix must be zero, so every antisymmetric matrix is of the form

$$\begin{pmatrix} 0 & a_{12} & a_{13}\\ -a_{12} & 0 & a_{23}\\ -a_{13} & -a_{23} & 0\end{pmatrix}.$$

Every matrix in the listed set is antisymmetric, so we need to see if the following set of equations has a unique solution:

$$\begin{aligned} a_{12} &= -c_1 + 3c_2 + 2c_3,\\ a_{13} &= 2c_1 - c_2 + c_3,\\ a_{23} &= 3c_1 + 2c_2 + c_3.\end{aligned}$$

Since there are three equations to determine the three constants, from elementary matrix theory we know that there will be a unique solution for all possible matrices if there is a unique solution for the zero matrix:

$$\begin{aligned} -c_1 + 3c_2 + 2c_3 &= 0,\\ 2c_1 - c_2 + c_3 &= 0,\\ 3c_1 + 2c_2 + c_3 &= 0.\end{aligned} \implies \begin{aligned} 5c_2 + 5c_3 &= 0,\\ 11c_2 + 7c_3 &= 0,\end{aligned} \implies c_1 = c_2 = c_3 = 0,$$

and hence the set is a basis.

(b) $\{e^x, \cosh x, 3\}$, solutions of $y^{(3)} - y' = 0$

Solution. We solve the equation by letting $y = e^{\lambda x}$ to obtain

$$\lambda^3 - \lambda = \lambda(\lambda^2 - 1) = \lambda(\lambda + 1)(\lambda - 1) = 0.$$

Thus the general solution is given by

$$y = c_1 e^x + c_2 e^{-x} + c_3 = (c_1 - c_2)e^x + c_2(e^x + e^{-x}) + c_3 = d_1 e^x + d_2 \cosh x + c_3.$$

Thus the set spans the space. We know that $y \equiv 0$ is a solution of the equation. Plugging $x = 0, x = \pm 1$ into the above, we have

$$\begin{aligned} d_1 + d_2 + c_3 &= 0 \\ d_1 e + d_2 \cosh 1 + c_3 &= 0 \\ d_1 e^{-1} + d_2 \cosh 1 + c_3 &= 0. \end{aligned}$$

Solving the last two equations together gives $d_1 = 0$. Solving the first two equations together with $d_1 = 0$ gives $d_2 = c_3 = 0$, so the set is linearly independent.

(c) $\{(1, 0, 1), (-2, 0, 3), (0, 1, 1)\}$, $\quad \mathcal{R}^3$

Solution. For this to be a basis of $\mathcal{R}^3$, we need to see if the following set of equations has a unique solution:

$$\begin{aligned} x_1 &= c_1 - 2c_2, \\ x_2 &= c_3, \\ x_3 &= c_1 + 3c_2. \end{aligned} \quad \Longrightarrow \quad c_2 = \frac{x_3 - x_1}{5}, \quad c_1 = \frac{3x_1 + 2x_3}{5},$$

and hence the set is a basis.

8. (5 points) Let $B_n = \{p_i(x)\}_0^n$ be a set of polynomials in $\mathcal{P}_n$, where p_i has degree i. Show that B_n is a basis for $\mathcal{P}_n$.

Solution. We proceed by induction. Clearly any constant polynomial is a basis for $\mathcal{P}_0$. Suppose B_{n-1} is a basis for $\mathcal{P}_{n-1}$. First, we check linear independence. If

$$\sum_{i=0}^{n} c_i p_i(x) = 0,$$

we see that $c_n = 0$ because all the other polynomials have degree less than n, so our check reduces to

$$\sum_{i=0}^{n-1} c_i p_i(x) = 0,$$

which by B_{n-1} being a basis must imply that all the $c_i = 0$ for $i < n$. Thus the set is linearly independent. Checking the spanning, we have

$$\operatorname{Span} B_n = \sum_{i=0}^{n} c_i p_i(x) = c_n p_n(x) + \sum_{i=0}^{n-1} c_i p_i(x).$$

The first term takes care of all possible coefficients of the x^n term. By B_{n-1} being a basis for $\mathcal{P}_{n-1}$, all possible coefficients of the lower-order terms are covered, so B_n is a basis for $\mathcal{P}_n$.