Homework Set 10 Solutions

1. (6 points) Suppose U is a finite-dimensional real vector space and $T \in \mathcal{L}(U)$. Prove that U has a basis consisting of eigenvectors of T if and only if there is an inner product on U that makes T self-adjoint.

Solution. If there is an inner product on U that makes T self-adjoint, then U is a real inner-product space and the result follows trivially from the Real Spectral Theorem. Now suppose that U has a basis $Z = \{\mathbf{z}_j\}$ consisting of eigenvectors of T. Let

$$\mathbf{v} = \sum_{j=0}^{n} c_j \mathbf{z}_j, \quad \mathbf{w} = \sum_{j=0}^{n} d_j \mathbf{z}_j,$$

and let us construct the following function:

$$F(\mathbf{v}, \mathbf{w}) = \sum_{j=0}^{n} c_j d_j = [\mathbf{v}]_Z \cdot [\mathbf{w}]_Z.$$

We know that the transformation $T \in \mathcal{L}(V, \mathcal{R}^n)$, $T(\mathbf{v}) = [\mathbf{v}]_Z$, is invertible. Since the dot product in $\mathcal{R}^n$ is an inner product, so is $F(\mathbf{v}, \mathbf{w})$ by Homework Set 7, #1(b). But then

$$F(T\mathbf{v}, \mathbf{w}) = F\left(\sum_{j=0}^{n} c_j T(\mathbf{z}_j), \sum_{j=0}^{n} d_j \mathbf{z}_j\right) = F\left(\sum_{j=0}^{n} \lambda_j c_j \mathbf{z}_j, \sum_{j=0}^{n} d_j \mathbf{z}_j\right) = \sum_{j=0}^{n} (\lambda_j c_j) d_j$$

$$F(\mathbf{v}, T^*\mathbf{w}) = \sum_{j=0}^{n} c_j (\lambda_j d_j) = F\left(\sum_{j=0}^{n} c_j \mathbf{z}_j, \sum_{j=0}^{n} \lambda_j d_j \mathbf{z}_j\right) = F\left(\sum_{j=0}^{n} c_j \mathbf{z}_j, \sum_{j=0}^{n} d_j T(\mathbf{z}_j)\right)$$
$$= F(\mathbf{v}, T\mathbf{w}),$$

so T is self-adjoint.

2. Let V be a complex vector space, $T \in \mathcal{L}(V)$ be normal, W_i be the eigenspace of T corresponding to λ_i, $1 \leq i \leq m$, where each of the λ_i is distinct.

   (a) (4 points) Show that

$$T(\mathbf{v}) = \sum_{i=1}^{m} \lambda_i \mathrm{proj}_{W_i} \mathbf{v}.$$

Solution. By hypothesis, there exists an orthonormal basis $Z = \{\mathbf{z}_j\}_1^n$ for V consisting of eigenvectors of T. Let

$$\mathbf{v} = \sum_{j=1}^{n} c_j \mathbf{z}_j.$$

Then

$$\begin{aligned}T(\mathbf{v}) =& T\left(\sum_{j=1}^{n} c_j \mathbf{z}_j\right) = \sum_{j=1}^{n} \langle \mathbf{z}_j, \mathbf{v}\rangle T(\mathbf{z}_j) = \sum_{i=1}^{m} \lambda_i \sum_{T(\mathbf{z}_j)=\lambda_i \mathbf{z}_j} \langle \mathbf{z}_j, \mathbf{v}\rangle \mathbf{z}_j \\ =& \sum_{i=1}^{m} \lambda_i \sum_{\mathbf{z}_j \in W_i} \langle \mathbf{z}_j, \mathbf{v}\rangle \mathbf{z}_j = \sum_{i=1}^{m} \lambda_i \mathrm{proj}_{W_i} \mathbf{v},\end{aligned}$$

where in the last line we have used the fact that W_i is an invariant subspace of V under T, so the set of $\mathbf{z}_j \in W_i$ form an orthonormal basis for W_i.

(b) (3 points) Show that $\mathrm{proj}_{W_i} = g_i(T)$ for some polynomial g_i.

Solution. First, we note that

$$\begin{aligned}T^k(\mathbf{v}) =& T^k\left(\sum_{j=1}^{n} c_j \mathbf{z}_j\right) = \sum_{j=1}^{n} \langle \mathbf{z}_j, \mathbf{v}\rangle T^k(\mathbf{z}_j) = \sum_{i=1}^{m} \lambda_i^k \mathrm{proj}_{W_i} \mathbf{v}, \\ g(T) =& \sum_{k=0}^{n} a_k T^k = \sum_{k=0}^{n} a_k \sum_{i=1}^{m} \lambda_i^k \mathrm{proj}_{W_i} = \sum_{i=1}^{m} g(\lambda_i) \mathrm{proj}_{W_i}.\end{aligned}$$

As written, we can specify $n+1$ points $g(\lambda_j)$ and still have a well-defined polynomial. Thus as long as $n \geq m-1$, we can create a polynomial $g_i(\lambda)$ such that $g_i(\lambda_k) = \delta_{ik}$. Then we would have

$$g_i(T) = \sum_{k=1}^{m} g_i(\lambda_k) \mathrm{proj}_{W_k} = \mathrm{proj}_{W_i}.$$

3. (7 points) Let $T \in \mathcal{L}(V)$ be normal, dim $V = n$, $T\mathbf{z}_j = \lambda_j \mathbf{z}_j$, where the $\mathbf{z}_j$ are orthonormal and $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$. Define the *Rayleigh quotient*

$$R(\mathbf{v}) = \frac{\langle T\mathbf{v}, \mathbf{v}\rangle}{\|\mathbf{v}\|^2}.$$

Show that $\lambda_1 \leq R(\mathbf{v}) \leq \lambda_n$. For what values (if any) of $\mathbf{v}$ does equality hold? (Consider each inequality separately.)

Solution. Let

$$\mathbf{v} = \sum_{j=1}^{n} c_j \mathbf{z}_j.$$

Then

$$\begin{aligned}R(\mathbf{v}) = \frac{\langle T\mathbf{v}, \mathbf{v}\rangle}{\|\mathbf{v}\|^2} =& \left(\sum_{j=1}^{n} |c_j|^2\right)^{-1} \left\langle \sum_{j=1}^{n} c_j T(\mathbf{v}_j), \sum_{k=1}^{n} c_k \mathbf{v}_k \right\rangle \\ =& \left(\sum_{j=1}^{n} |c_j|^2\right)^{-1} \sum_{j=1}^{n} \overline{c_j} \sum_{k=1}^{n} c_k \langle \lambda_j \mathbf{v}_j, \mathbf{v}_k\rangle = \left(\sum_{j=1}^{n} |c_j|^2\right)^{-1} \sum_{j=1}^{n} \lambda_j |c_j|^2.\end{aligned}$$

Writing in a convenient form for the first inequality, we have

$$R(\mathbf{v}) = \lambda_1 \left(\sum_{j=1}^{n} |c_j|^2\right)^{-1} \left(\sum_{j=1}^{n} \frac{\lambda_j}{\lambda_n}|c_j|^2\right).$$

But $\lambda_1 \leq \lambda_j$, so each of the fractions is greater than or equal to 1. Thus each summand in the numerator is larger than the corresponding summand in the denominator, so $R(\mathbf{v}) \geq \lambda_1$. Equality holds when $c_j = 0$ for $\lambda_j \neq \lambda_1$, so $\mathbf{v}$ is in the eigenspace corresponding to λ_1. Similarly,

$$R(\mathbf{v}) = \lambda_n \left(\sum_{j=1}^{n} |c_j|^2\right)^{-1} \left(\sum_{j=1}^{n} \frac{\lambda_j}{\lambda_n}|c_j|^2\right).$$

But $\lambda_n \geq \lambda_j$, so each of the fractions is less than or equal to 1. Thus each summand in the numerator is smaller than the corresponding summand in the denominator, so $R(\mathbf{v}) \leq \lambda_1$. Equality holds when $c_j = 0$ for $\lambda_j \neq \lambda_n$, so $\mathbf{v}$ is in the eigenspace corresponding to λ_n.

4. Find the conditions (if any) under which each of the linear transformations are isometries.
   (a) (3 points) Given a particular $\mathbf{u} \in \mathcal{C}^n$, consider $S \in \mathcal{L}(\mathcal{C}^n)$ defined by $S = I - 2\mathbf{u}\mathbf{u}^*$.

Solution. Let $\mathbf{v} \in \mathcal{C}^n$, $\|\mathbf{v}\|^2 = \mathbf{v}^*\mathbf{v}$. Then

$$\begin{aligned}\|S\mathbf{v}\|^2 =&\|\mathbf{v} - 2\mathbf{u}\mathbf{u}^*\mathbf{v}\|^2 = (\mathbf{v} - 2(\mathbf{u}^*\mathbf{v})\mathbf{u})^*(\mathbf{v} - 2(\mathbf{u}^*\mathbf{v})\mathbf{u}) \\ =&\mathbf{v}^*\mathbf{v} - 2(\mathbf{u}^*\mathbf{v})\mathbf{u}^*\mathbf{v} - 2(\mathbf{u}^*\mathbf{v})\mathbf{v}^*\mathbf{u} + 4(\mathbf{u}^*\mathbf{v})(\mathbf{u}^*\mathbf{v})\mathbf{u}^*\mathbf{u} \\ =&\|\mathbf{v}\|^2 - 4|\mathbf{u}^*\mathbf{v}|^2 + 4|\mathbf{u}^*\mathbf{v}|^2\|\mathbf{u}\|^2 \\ =&\|\mathbf{v}\|^2 + 4(\|\mathbf{u}\|^2 - 1)|\mathbf{u}^*\mathbf{v}|^2.\end{aligned}$$

Thus for S to be an isometry, $\|\mathbf{u}\|^2 = 1$ (to make the parenthetical term zero) or $\mathbf{u} = \mathbf{0}$ (to make the inner product zero).

   (b) (4 points) Given a complex-valued polynomial $h(t) \in C[0, 1]$, let $S(f) = hf$. Use the inner product

$$\langle f, g\rangle = \int_0^1 \bar{f}g\, dt.$$

Solution.

$$\|S(f)\|^2 - \|f\|^2 = \int_0^1 \overline{(hf)}hf - \bar{f}f\, dt = \int_0^1 (|h|^2 - 1)\bar{f}f\, dt.$$

For this to be true for every $f, |h(t)|^2 = 1$ for all $t \in [0, 1]$.

5. (2 points) Let $S \in \mathcal{L}(V)$ be an isometry, $\mathbf{u}$, $\mathbf{v} \in V$. Show that the angle between $\mathbf{u}$ and $\mathbf{v}$ is the same as the angle between $S\mathbf{u}$ and S$\mathbf{v}$.

Solution. Using the definition of the angle, we have

$$\text{angle between } S\mathbf{u} \text{ and } S\mathbf{v} = \frac{\langle S\mathbf{u},\, S\mathbf{v}\rangle}{\|S\mathbf{u}\|^2\|S\mathbf{v}\|^2} = \frac{\langle \mathbf{u},\, \mathbf{v}\rangle}{\|\mathbf{u}\|^2\|\mathbf{v}\|^2} = \text{angle between } \mathbf{u} \text{ and } \mathbf{v},$$

where we have used properties of isometries.

6. Let $T \in \mathcal{L}(V)$, dim $V = n$. Let $\mathbf{z}_0 = \mathbf{0}$ and define $\mathbf{z}_j$, $j > 0$ by the following recurrence relation:

$$(T - \lambda I)\mathbf{z}_j = \mathbf{z}_{j-1}. \tag{1}$$

(a) (3 points) Show that $\mathbf{z}_j$ (if defined) is an eigenvector for T and λ of order j.

Solution. Proof by induction on j. For $j = 1$, we have $(T - \lambda I)\mathbf{z}_1 = \mathbf{0}$, which implies that $\mathbf{z}_1$ is an eigenvector for T and λ, which means it is a generalized eigenvector of order 1. Assume true for $j-1$, prove for j. In this case, we have $(T-\lambda I)\mathbf{z}_j = \mathbf{z}_{j-1}$. Since $\mathbf{z}_{j-1}$ is an eigenvector of order $j-1$, we have that $(T-\lambda I)^j\mathbf{z}_j = (T-\lambda I)^{j-1}\mathbf{z}_{j-1} = \mathbf{0}$, so $\mathbf{z}_j$ is a generalized eigenvector of order no larger than j. But since $\mathbf{z}_{j-1}$ is an eigenvector of order $j-1$, $(T-\lambda I)^k\mathbf{z}_{j-1} \neq \mathbf{0}$ for $k < j-1$, so $(T-\lambda I)^k\mathbf{z}_{j-1} \neq \mathbf{0}$ for $k < j$.

(b) (2 points) Show that there will be no solution $\mathbf{z}_j$ to (1) for $j > m$, where $m \leq n$.

Solution. By part (a), equation (1) is equivalent to solving

$$\mathbf{z}_j \in \mathcal{N}((T-\lambda I)^j), \qquad \mathbf{z}_j \notin \mathcal{N}((T-\lambda I)^{j-1}). \tag{A}$$

But by Proposition 8.5, there exists a non-negative integer m at most n such that $\mathcal{N}((T-\lambda I)^m) = \mathcal{N}((T-\lambda I)^{m+1})$. Therefore, there would be no solution of (A) for $j \geq m+1$.

7. For each of the following operators, find an eigenvector and generalized eigenvectors for each distinct order corresponding to the sole eigenvalue.

(a) (3 points) $V = \mathcal{R}^2$, $T(x_1, x_2) = (x_1 + 2x_2, -2x_1 + 5x_2)$.

Solution. Solving for the eigenvalue and eigenvector, we have

$$\begin{aligned} T(x_1,x_2) &= \lambda(x_1,x_2) \\ (1-\lambda)x_1 + 2x_2 &= 0 \\ -2x_1 + (5-\lambda)x_2 &= 0 \\ (1-\lambda)(5-\lambda) + 4 = \lambda^2 - 6\lambda + 9 = (\lambda-3)^2 &= 0 \\ -2x_1 + 2x_2 &= 0 \\ \mathbf{z}_1 &= (2,2). \end{aligned}$$

Using our result from (1), we know that for the generalized eigenvector, we must solve

$$\begin{aligned} (T-3I)\mathbf{z}_2 &= \mathbf{z}_1 \\ -2x_1 + 2x_2 &= 2 \\ \mathbf{z}_2 &= (0,1). \end{aligned}$$

(b) (3 points) $V = \mathcal{P}_2$, $T(p(x)) = 2p - p'$.

Solution. Solving for the eigenvalue and eigenvector, we have

$$\begin{aligned} T(a_0 + a_1 x + a_2 x^2) = 2(a_0 + a_1 x + a_2 x^2) - (a_1 + 2a_2 x) &= \lambda(a_0 + a_1 x + a_2 x^2) \\ \lambda &= 2 \qquad\qquad O(x^2) \\ (a_1 + 2a_2 x) &= 0 \\ p_1 &= 2. \end{aligned}$$

Since we found $\lambda = 2$ by assuming that $a_2 \neq 0$, we rework with $a_2 = 0$ to see if we obtain another eigenvalue:

$$\begin{aligned} T(a_0 + a_1 x) = 2(a_0 + a_1 x) - (a_1) &= \lambda(a_0 + a_1 x) \\ \lambda &= 2, \qquad\qquad O(x) \end{aligned}$$

and hence we see that $\lambda = 2$ is our only eigenvalue. Using our result from (1), we know that for the generalized eigenvectors, we must solve

$$\begin{aligned} (T - 2I)p_2 &= -p_2' = p_1 = 2 \\ p_2 &= -2x \\ (T - 2I)p_3 &= -p_3' = p_2 = -2x \\ p_3 &= x^2. \end{aligned}$$