Homework Set 1 Solutions

1. (2 points) Prove that the multiplicative identity for a vector space is unique.

Solution. Let a_1 and a_2 be two real scalars such that

$$a_1\mathbf{v} = \mathbf{v}, \quad a_2\mathbf{v} = \mathbf{v},$$

for any $\mathbf{v} \in V$, the vector space in question. Subtracting, we obtain

$$a_1\mathbf{v} - a_2\mathbf{v} = (a_1 - a_2)\mathbf{v} = \mathbf{0},$$

where we have used the distributive law. Since this is true for any $\mathbf{v}, a_1 - a_2 = 0$, and since this is subtraction of real numbers, $a_1 = a_2$.

2. (4 points per part) For the following sets, either prove the set is a vector space or find a property which the set violates.

   (a) Vectors: solutions to the differential equation $y'' + y = 0$
   Vector addition: $f(t) + g(t) = (f + g)(t)$.
   Scalar multiplication: $c[f(t)] = (cf)(t)$

Solution. Let y_1 and y_2 be two solutions to the differential equation. We check the properties one by one. The only difficult ones are closure:

$$(y_1 + y_2)'' + (y_1 + y_2) = (y_1'' + y_1) + (y_2'' + y_2) = 0 + 0 = 0, \tag{A1}$$

$$(cy_1)'' + (cy_1) = c(y_1'' + y_1) = c(0) = 0. \tag{M1}$$

The association, commutation, and distribution properties hold because we are adding and multiplying real-valued functions. The zero vector is simply the zero function, which trivially satisfies $y'' + y = 0$. The multiplicative identity is just 1, since we are multiplying real-valued functions. The additive inverse is just $-f(t)$. Since this is a multiple of $f(t)$, by M1 it is in the vector space.

   (b) Vectors in $\mathcal{C}^2$
   Vector addition: $(x_1, x_2) + (y_1, y_2) = (x_1 + 3y_1, x_2 + 2y_2)$
   Scalar multiplication: $c(x_1, x_2) = (cx_1, cx_2)$

Solution. This is not a subspace; one reason is because addition is not commutative:

$$(x_1, x_2) + (y_1, y_2) = (x_1 + 3y_1, x_2 + 2y_2)$$

$$(y_1, y_2) + (x_1, x_2) = (y_1 + 3x_1, y_2 + 2x_2).$$

In general, these two vectors are not equal.

3. (5 points) Prove that the only subspaces of $\mathcal{R}^2$ are $\{\mathbf{0}\}$, $\mathcal{R}^2$, and subspaces of the form $W_{\mathbf{v}} = \{c\mathbf{v},\ c \in \mathcal{R}\}$ for some $\mathbf{v} \in \mathcal{R}^2$.

Solution. Clearly $\mathcal{R}^2$ is a subspace of $\mathcal{R}^2$. $\{\mathbf{0}\}$ is a special case of $W_{\mathbf{v}}$ with $\mathbf{v} = \mathbf{0}$, so we check if $W_{\mathbf{v}}$ is a subspace. Let $\mathbf{w}_1 = c_1\mathbf{v}$ and $\mathbf{w}_2 = c_2\mathbf{v}$ be two vectors in $W_{\mathbf{v}}$. Then we have

$$\begin{aligned} \mathbf{w}_1 + \mathbf{w}_2 = c_1\mathbf{v} + c_2\mathbf{v} &= (c_1 + c_2)\mathbf{v} \in W_{\mathbf{v}}, && \text{(A1)} \\ c\mathbf{w}_1 = c(c_1\mathbf{v}) &= (cc_1)\mathbf{v} \in W_{\mathbf{v}}, && \text{(M1)} \\ \mathbf{0} = 0\mathbf{w}_1 &= 0\mathbf{v} \in W_{\mathbf{v}}. && \text{(A3)} \end{aligned}$$

Therefore $W_{\mathbf{v}}$ is a subspace of $\mathcal{R}^2$. Now assume that there exists a subspace V not of the form $W_{\mathbf{v}}$. It must contain at least two vectors $\mathbf{v}$ and $\mathbf{w}$, where $\mathbf{w} \neq c\mathbf{v}$. Then verifying property (M1), we must have that $c\mathbf{v} + d\mathbf{w} \in V$ for any real constants c and d. This means that vectors in V are generated by two noncollinear vectors with arbitrary constants. But this is exactly $\mathcal{R}^2$.

4. (2 points per part) Consider the vector space $\mathcal{C}(\mathcal{R})$ (the space of all real–valued continuous functions defined for all $\mathcal{R}$). For each of the following subsets of $\mathcal{C}(\mathcal{R})$, either prove the set is a subspace of $\mathcal{C}(\mathcal{R})$ or find a property which the set violates.

(a) All non-increasing functions.

Solution. This is not a subspace because it violates property (M1). To wit, let f be a decreasing function. If the set of all decreasing functions is a subspace, $-f$ must be a decreasing function as well. But if f decreases, $-f$ increases.

(b) All differentiable functions.

Solution. Denote the set of all differentiable functions by W, and let $f_1, f_2 \in W$. Then checking the properties, we obtain

$$\begin{aligned} f_1' + f_2' = (f_1 + f_2)' &\implies f_1 + f_2 \in W, && \text{(A1)} \\ c(f_1') = (cf_1)' &\implies cf_1 \in W, && \text{(M1)} \\ 0' = 0 &\implies 0 \in W. && \text{(A3)} \end{aligned}$$

Therefore W is a subspace of $\mathcal{C}(\mathcal{R})$.

(c) All integrable functions.

Solution. Denote the set of all integrable functions by W, and let $f_1, f_2 \in W$. Then checking the properties, we obtain

$$\begin{aligned} \int_{-\infty}^{\infty} f_1(x)\,dx + \int_{-\infty}^{\infty} f_2(x)\,dx = \int_{-\infty}^{\infty} (f_1 + f_2)(x)\,dx &\implies f_1 + f_2 \in W, && \text{(A1)} \\ c\int_{-\infty}^{\infty} f_1(x)\,dx = \int_{-\infty}^{\infty} (cf_1)(x)\,dx &\implies cf_1 \in W, && \text{(M1)} \\ \int_{-\infty}^{\infty} 0\,dx = 0 &\implies 0 \in W, && \text{(A3)} \end{aligned}$$

where we have used the fact that all of the integrals on the left-hand side are bounded. Therefore W is a subspace of $\mathcal{C}(\mathcal{R})$.

5. Let M_+ be the set of all symmetric matrices in $\mathcal{R}^{n\times n}$, and M_- be the set of all antisymmetric matrices in $\mathcal{R}^{n\times n}$.

(a) (3 points) Verify that M_- is a subspace of $\mathcal{R}^{n\times n}$.

Solution. Let $A, B \in M_-$, so $A^T = -A$ and $B^T = -B$. Then checking the properties, we see that

$$(A+B)^T = A^T + B^T = -A - B = -(A+B) \quad \Longrightarrow \quad A+B \in M_-, \qquad \text{(A1)}$$
$$(cA)^T = cA^T = c(-A) = -(cA) = \quad \Longrightarrow \quad cA \in M_-, \qquad \text{(M1)}$$
$$O^T = O = -O \quad \Longrightarrow \quad 0 \in M_-. \qquad \text{(A3)}$$

Therefore M_- is a subspace of $\mathcal{R}^{n\times n}$.

(b) (6 points) Show that $\mathcal{R}^{n\times n} = M_+ \oplus M_-$.

Solution. We use Proposition 1.9. First, we show that $\mathcal{R}^{n\times n} = M_+ + M_-$ by showing one way in which any matrix $A \in \mathcal{R}^{n\times n}$ can be written as the sum of a symmetric matrix and an antisymmetric matrix. Rewriting

$$A = \frac{A_+ + A_-}{2}, \quad A_+ = A + A^T,\ A_- = A - A^T,$$

we note that

$$\begin{aligned} A_+^T &= (A + A^T)^T = A^T + A = A_+, \qquad A_+ \in M_+, \\ A_-^T &= (A - A^T)^T = A^T - A = -A_-, \qquad A_- \in M_- \end{aligned}$$

To show that it is a direct sum, we verify that the only matrix that is both symmetric and antisymmetric is the zero matrix. Call this matrix Z. Then we know that

$$Z = Z^T = -Z,$$

which implies that $2Z = O$, so Z is the zero matrix.

6. Let $\mathcal{C}_{\mathrm{e}}(\mathcal{R})$ be the set of all even continuous functions defined on $\mathcal{R}$, and let $\mathcal{C}_{\mathrm{o}}(\mathcal{R})$ be the set of all odd continuous functions defined on $\mathcal{R}$.

(a) (4 points) Prove that $\mathcal{C}_{\mathrm{e}}(\mathcal{R})$ and $\mathcal{C}_{\mathrm{o}}(\mathcal{R})$ are subspaces of $\mathcal{C}(\mathcal{R})$.

Solution. Let $f_1,\ f_2 \in \mathcal{C}_{\mathrm{e}}(\mathcal{R})$. Then checking the properties, we obtain

$$(f_1 + f_2)(-x) = f_1(-x) + f_2(-x) = f_1(x) + f_2(x)$$
$$(f_1 + f_2)(-x) = (f_1 + f_2)(x) \quad \Longrightarrow \quad f_1 + f_2 \in \mathcal{C}_{\mathrm{e}}(\mathcal{R}), \quad \text{(A1)}$$
$$(cf_1)(-x) = c(f_1(-x)) = c(f_1(x)) = (cf_1)(x) \quad \Longrightarrow \quad cf_1 \in \mathcal{C}_{\mathrm{e}}(\mathcal{R}), \qquad \text{(M1)}$$
$$0(-x) = 0 = 0(x) \quad \Longrightarrow \quad 0 \in \mathcal{C}_{\mathrm{e}}(\mathcal{R}). \qquad \text{(A3)}$$

Therefore $\mathcal{C}_{\mathrm{e}}(\mathcal{R})$ is a subspace of $\mathcal{C}(\mathcal{R})$. Let $g_1, g_2 \in \mathcal{C}_{\mathrm{o}}(\mathcal{R})$. Then checking the properties, we obtain

$$(g_1 + g_2)(-x) = g_1(-x) + g_2(-x) = -g_1(x) - g_2(x)$$
$$(g_1 + g_2)(-x) = -(g_1 + g_2)(x) \quad \Longrightarrow \quad g_1 + g_2 \in \mathcal{C}_{\mathrm{o}}(\mathcal{R}), \ \text{(A1)}$$
$$(cg_1)(-x) = c(g_1(-x)) = c(-g_1(x)) = -(cg_1)(x) \quad \Longrightarrow \quad cg_1 \in \mathcal{C}_{\mathrm{o}}(\mathcal{R}), \qquad \text{(M1)}$$
$$0(-x) = 0 = -0(x) \quad \Longrightarrow \quad 0 \in \mathcal{C}_{\mathrm{o}}(\mathcal{R}). \qquad \text{(A3)}$$

Therefore $\mathcal{C}_{\mathrm{o}}(\mathcal{R})$ is a subspace of $\mathcal{C}(\mathcal{R})$.

(b) (6 points) Using the *definition* of the direct sum, prove that $\mathcal{C}(\mathcal{R}) = \mathcal{C}_{\mathrm{e}}(\mathcal{R}) \oplus \mathcal{C}_{\mathrm{o}}(\mathcal{R})$.

Solution. To use the definition, we must show that any $f \in \mathcal{C}(\mathcal{R})$ can be written uniquely as the sum of $g_{\mathrm{e}} \in \mathcal{C}_{\mathrm{e}}(\mathcal{R})$ and $g_{\mathrm{o}} \in \mathcal{C}_{\mathrm{o}}(\mathcal{R})$. First we show that such a construction exists. Let

$$g_{\mathrm{e}}(x) = \frac{f(x) + f(-x)}{2}, \quad g_{\mathrm{o}}(x) = \frac{f(x) - f(-x)}{2}.$$

Clearly $f(x) = g_{\mathrm{e}}(x) + g_{\mathrm{o}}(x)$. We also note that

$$g_{\mathrm{e}}(-x) = \frac{f(-x) + f(x)}{2} = g_{\mathrm{e}}(x) \Longrightarrow g_{\mathrm{e}} \in \mathcal{C}_{\mathrm{e}}(\mathcal{R}),$$
$$g_{\mathrm{o}}(-x) = \frac{f(-x) - f(x)}{2} = -g_{\mathrm{o}}(x) \Longrightarrow g_{o}(x) \in \mathcal{C}_{\mathrm{o}}(\mathcal{R}).$$

Now we must show that g_{e} and g_{o} are unique. Suppose that there exist $h_{e} \in \mathcal{C}_{\mathrm{e}}(\mathcal{R})$ and $h_{\mathrm{o}} \in \mathcal{C}_{\mathrm{o}}(\mathcal{R})$ such that $f(x) = h_{\mathrm{e}}(x) + h_{\mathrm{o}}(x)$. Subtracting the two equations, we have

$$\begin{aligned} 0 &= [g_{\mathrm{e}}(x) - h_{\mathrm{e}}(x)] + [g_{\mathrm{o}}(x) - h_{\mathrm{o}}(x)] \\ h_{\mathrm{e}}(x) - g_{\mathrm{e}}(x) &= g_{\mathrm{o}}(x) - h_{\mathrm{o}}(x). \end{aligned}$$

Now the left-hand side of the equation is an even function, while the right-hand side is an odd function. Call this function $z(x)$. Then we know that for all x,

$$z(x) = z(-x) = -z(x) \quad \Longrightarrow \quad z(x) = 0,$$

so

$$h_{\mathrm{e}}(x) = g_{\mathrm{e}}(x), \quad g_{\mathrm{o}}(x) = h_{\mathrm{o}}(x),$$

and the factorization is unique. (Note that we have essentially proven Propositions 1.8 and 1.9 for this special case.)