Homework Set 8 (Revised)

Read pp. 108–118. For the inner product problems,

The Gram–Schmidt Process

1. (5 points per part)
   (a) page 123, number 10
   (b) Repeat part (a) using the inner product given by

$$\langle p, q\rangle = \int_{-\infty}^{\infty} p(x)q(x)e^{-x^2}\,dx.$$

   You should obtain

$$W = \left\{\pi^{-1/4},\ \frac{x\sqrt{2}}{\pi^{1/4}},\ \frac{\sqrt{2}}{\pi^{1/4}}\left(x^2 - \frac{1}{2}\right)\right\}.$$

2. (4 points) Determine an orthonormal basis for Span S, where

$$S = \{(4, -2, 2, -1), (-8, 6, -3, 0), (7, 5, 2, -3)\}.$$

3. (3 points) Let V be an inner product space, $W = \{\mathbf{w}_j\}_1^n$ be an orthonormal basis for V, and $T \in \mathcal{L}(V)$. Show that

$$[\mathcal{M}(T, W, W)]_{ij} = \langle \mathbf{w}_i, T(\mathbf{w}_j)\rangle.$$

Minimization Problems

4. For each of the following cases, find the vector $\mathbf{w} \in$ Span W that is closest to $\mathbf{v}$, given the inner product shown.
   (a) (7 points) $\mathbf{v} = x^2, W = \{1, \sin x, \cos x, \sin 2x, \cos 2x\}$,

$$\langle f, g\rangle = \int_{-\pi}^{\pi} f(x)g(x)\,dx.$$

   (b) (5 points) $\mathbf{v} = \sin \pi x,\ W = \{1, x, x^2\}$,

$$\langle f, g\rangle = \int_{0}^{1} f(x)g(x)\,dx.$$

Linear Functionals

5. (3 points) page 125, number 24
6. (5 points) Consider $L^2[a,b]$ to be an inner product space with inner product

$$\langle f_1, f_2 \rangle = \int_a^b f_1 f_2 \, dx,$$

and let U be a finite-dimensional subspace of $L^2[a,b]$. Given some $f \in L^2[a,b]$, consider the linear functional $T \in \mathcal{L}(U, \mathcal{R})$ given by

$$T(u) = \int_a^b fu \, dx.$$

Find a vector $g \in U$ such that $T(u) = \langle u, g \rangle$.

7. (3 points) Let A, $B \in \mathcal{R}^{n \times n}$,

$$\phi(A) = \sum_{i=1}^{n} \sum_{j=1}^{n} (-1)^{i+j} a_{ij}.$$

Find a matrix B such that $\phi(A) = \langle A, B \rangle$, where the inner product is as defined in Homework Set 7, #2(a).