Updates

1. The midterm will be administered on Friday, Oct. 24 during the normal class session. You will need a small blue examination book.
2. Because of the intervening exam, this homework set is not due until Nov. 3. However, you are encouraged to do the first homework question before Friday to help you prepare for the exam.

Homework Set 7 (Revised)

Read pp. 81–87, 97–108, 111-116.

Upper Triangular Matrices

1.
   (a) (2 points) page 95, number 18
   (b) (2 points) page 95, number 19

Inner Product Spaces

2.
   (a) (3 points) Recall that the *trace* of a matrix $A \in \mathcal{R}^{n\times n}$, denoted tr A, is defined as

$$\operatorname{tr} A = \sum_{i=1}^{n} a_{ii}.$$

   Let $B \in \mathcal{R}^{n\times n}$. Prove or disprove that the function $\langle A, B\rangle = \operatorname{tr}(B^T A)$ is an inner product on $\mathcal{R}^{n\times n}$.

   (b) (4 points) Let $T \in \mathcal{L}(V, W)$, V be a vector space over a field F, W be an inner-product space over F with a defined inner product $P_W(\mathbf{w}_1, \mathbf{w}_2)$ for $\mathbf{w}_i \in W$. Find conditions on T, if any, such that $P_V(\mathbf{v}_1, \mathbf{v}_2) = P_W(T(\mathbf{v}_1), T(\mathbf{v}_2))$ is an inner product on V.

Norms

3. For each of the following candidates, prove that the function is a norm on the given vector space V (listing any necessary conditions), or show that it violates some required property.

   (a) (2 points) $\qquad V = C[0,1],\ \|f\| = \int_0^1 |f(t)|\,dt$

   (b) (3 points) $\qquad V = \mathcal{P}_n,\ \|p\| = \sum_{i=0}^{k} p^2(x_i)$

   (c) (2 points) $\qquad V = \mathcal{R}^{n\times n},\ \|A\| = \max_{i,j} |a_{ij}|$

4. Consider the following candidates for vector norms:

$$\|\mathbf{x}\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}, \quad p \geq 1; \quad \|\mathbf{x}\|_\infty = \max_i |x_i|.$$

   (a) (5 points) Show that $\|\mathbf{x}\|_p$ is a vector norm by verifying the properties shown in class. (*Hint: You may use Hölder's Theorem without proof.*)

   (b) (2 points) Show that

$$\lim_{p\to\infty} \|\mathbf{x}\|_p = \|\mathbf{x}\|_\infty.$$

Angles and Orthogonality

5. (3 points) page 122, number 2
6. (4 points) page 123, number 9. This problem can be done cleverly without reference to integral tables. However, if you wish to do the integrals directly, you may use tables or Maple for the *indefinite* form as long as you provide a citation.

Orthogonal Projections

7. Let V be an inner product space and $W = \{\mathbf{w}_j\}_1^m$ be an orthonormal subset of V.

   (a) (3 points) Prove *Bessel's Inequality*:

$$\|\mathbf{x}\|^2 \geq \sum_{j=1}^{m} |\langle \mathbf{x}, \mathbf{w}_j\rangle|^2 \qquad \forall \mathbf{x} \in V.$$

   (b) (2 points) Prove that Bessel's Inequality becomes an equality if and only if $\mathbf{x} \in \text{Span } W$.

8. (3 points) Let U be a subspace of a finite–dimensional vector space V. Calculate $\text{proj}_{U^\perp}$ in terms of proj_U.