Homework Set 5

Read pp. 50–58, 76–79.

Matrices of Linear Transformations

1. (4 points) page 61, number 20
2. Consider the following transformations:

$$T_1 \in \mathcal{L}\left(P_2 \times \mathcal{R}, \mathcal{R}^2 \times P_1\right), T_1\left(p(x), y\right) = \left(y + p(1), p(0) - 2y, p'(x)\right)$$

$$T_2 \in \mathcal{L}(\mathcal{R}^2 \times P_1, \mathcal{R}^3), \quad T_2(z_1, z_2, p(x)) = \left(p'(1) - 3z_1, \int_0^1 xp(x)\,dx, z_1 + p(0)\right),$$

where the notation $V \times W$ indicates the vector space of ordered pairs where the first element is in V and the second is in W. Using the standard bases E in all cases, verify the following facts:

(a) (5 points) $\mathcal{M}\left(T_2T_1\right) = \mathcal{M}\left(T_2\right)\mathcal{M}\left(T_1\right)$

(b) (2 points) $\left[T_1\left(x^2 - 1, 3\right)\right]_E = \mathcal{M}\left(T_1\right)\left[\left(x^2 - 1, 3\right)\right]_E$

(c) (2 points) $\left[T_2T_1\left(x^2 - 1, 3\right)\right]_E = \mathcal{M}\left(T_2\right)\left[T_1\left(x^2 - 1, 3\right)\right]_E$

Invariant Subspaces and Eigenspaces

3.
(a) (5 points) page 95, number 11

(b) (3 points) Show that $ST - TS \neq I$ for any $S, T \in \mathcal{L}(V)$.

4. (4 points) Let $T \in \mathcal{L}(V)$, U a T-invariant subspace of V. Let $\dim U = m, \dim V = n$. Show that there exists a basis W for V such that

$$\mathcal{M}\left(T, W, W\right) = \begin{pmatrix} A & B \\ O & C \end{pmatrix},\ A \in \mathcal{R}^{m\times m},\ B \in \mathcal{R}^{m\times(n-m)},\ C \in \mathcal{R}^{(n-m)\times(n-m)}$$

and O is the zero matrix in $\mathcal{R}^{(n-m)\times m}$.

5. An operator $N \in \mathcal{L}(V)$ is called *nilpotent* if $N^k = 0$ for some integer k.
   (a) (3 points) Show that the only eigenvalue for N is $\lambda = 0$.
   (b) (3 points) Let $V = \mathcal{R}^n$. Give an example of a nilpotent operator N such that $N^n = 0$, but $N^j \neq 0$ for any $j < n$.
6. (3 points) Let $T \in \mathcal{L}(V)$ be an invertible transformation, and let U be a subspace of V. Prove that if U is invariant under T, U is invariant under T^{-1} .
7. Let $T \in \mathcal{L}(V)$,

$$V = \bigoplus_{i=1}^{n} U_i,$$

where U_i is invariant under T. Show that

(a) (3 points)

$$R(T) = \bigoplus_{i=1}^{n} R\left(T\big|_{U_i}\right)$$

(b) (3 points)

$$\mathcal{N}(T) = \bigoplus_{i=1}^{n} \mathcal{N}\left(T\big|_{U_i}\right)$$