MATH 672-010 | Vector Spaces
Prof. D. A. Edwards | Due: Oct. 6, 2008

Homework Set 4 (Revised)

Read chapter 3.

Properties of Linear Transformations

1. Let $T \in \mathcal{L}(V, W)$, $S_V = \{\mathbf{s}_i\}_1^n \in V$, $S_W = \{T(s_i)\}_1^n \in W$. For each of the following statements, provide a proof or counterexample.
   (a) (2 points) If S_V is linearly independent, so is S_W.
   (b) (2 points) If S_W is linearly independent, so is S_V.

Dimension

2. (5 points) page 60, number 16
3. For any polynomial $p(t) \in \mathcal{P}_3$, let $D(p) = p'$.
   (a) (2 points) Show that if we consider $D : \mathcal{P}_3 \to \mathcal{P}_2$, D is onto, but not one-to-one. In the process, calculate the dimensions necessary to verify Theorem 3.4.

   Now let $S = \{p \in \mathcal{P}_3 \,|\, p(1) = 0\}$.

   (b) (2 points) Show that if we consider $D : S \to \mathcal{P}_3$, D is one-to-one, but not onto. In the process, calculate the dimensions necessary to verify Theorem 3.4.
4. Consider the following transformation $T : \mathcal{R}^3 \to \mathcal{R}^3$:

$$T(x_1, x_2, x_3) = (x_2, x_2, x_2).$$

   (a) (2 points) Calculate $\mathcal{N}(T)$.
   (b) (2 points) Calculate $R(T)$.
   (c) (2 points) Verify Theorem 3.4.
   (d) (2 points) Is T one-to-one? Onto? Invertible?
5. Consider the following transformation $T \in \mathcal{L}(V, W)$:

$$f \in C^2(\mathcal{R}) : \quad T(f) = t^2 f'' - 6tf' + 12f.$$

   (a) (4 points) Let $V = W = C^2(\mathcal{R})$. Calculate $\mathcal{N}(T)$. Is T one-to-one? Invertible?
   (b) (3 points) Let $V = W = \mathcal{P}_2(\mathcal{R})$. Calculate $\mathcal{N}(T)$ and $R(T)$. Verify Theorem 3.4 and determine whether T is one-to-one, onto, or invertible.

Matrices of Linear Transformations

6. Let $B_1 = \{\cos x, \sin x\}$ and $B_2 = \{\cos x + \sin x, \cos x - \sin x\}$ be two bases for the same subspace V of $C^1(\mathcal{R})$, and let $D \in \mathcal{L}(V)$ be defined by $D(f) = f'$ for any $f \in V$. Compute the following:
   (a) (2 points) $\mathcal{M}(D, B_1, B_1)$
   (b) (2 points) $\mathcal{M}(D, B_1, B_2)$
   (c) (2 points) $\mathcal{M}(D, B_2, B_1)$
   (d) (2 points) $\mathcal{M}(D, B_2, B_2)$

7. Let $T : \mathcal{P}_2 \to \mathcal{R}^2$ be defined by

$$T(p(x)) = \begin{pmatrix} p(1) \\ \displaystyle\int_0^1 p(x)\,dx \end{pmatrix}.$$

   (a) (2 points) Calculate $\mathcal{M}(T)$ for the standard bases.
   (b) (2 points) Is T one-to-one? Onto? Invertible?