Homework Set 5 Solutions

1. (a) (5 points) Calculate $\Delta$ for a digital call.

**Solution.** From notes in class, we have that the value of a digital call is given by

$$V = \frac{V_2}{K} = e^{-r(T-t)}N(d_2), \quad d_2 = \frac{\log(S/K) + (r - \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}.$$ 

Therefore, we have

$$\Delta = \frac{\partial V}{\partial S} = e^{-r(T-t)} \frac{\partial N(d_2)}{\partial S} = e^{-r(T-t)}n(d_2) \frac{\partial d_2}{\partial S} = e^{-r(T-t)} \frac{n(d_2)}{S\sigma\sqrt{T-t}}.$$ 

(b) (4 points) Suppose that $t$ is near expiry and $S$ is near the strike. In particular, let $t = T - \epsilon^2$, $S = K(1 + k\epsilon)$, where $0 < \epsilon \ll 1$. What happens to $\Delta$ in this instance? Discuss the practicality of hedging such an option in this case.

**Solution.** Substituting these expressions into our equation for $d_2$, we have that

$$d_2 = \frac{\log(1 + k\epsilon) + (r - \sigma^2/2)\epsilon^2}{\sigma\epsilon} = \frac{k}{\sigma} + O(\epsilon)$$

and

$$\Delta \sim e^{-r\epsilon^2} \frac{n(k/\sigma)}{K(1 + k\epsilon)\sigma\epsilon} \sim \frac{n(k/\sigma)}{K\sigma\epsilon},$$

which gets arbitrarily large as $\epsilon \to 0$. Hence one would have to amass a very large number of shares to hedge, which is impractical.

2. (8 points) Show mathematically that the vega for a European put is the same as for a European call. Explain your result from a portfolio perspective.

**Solution.** From notes in class, we have that the value $V_p$ of the European put is given by

$$V_p = V_c + Ke^{-r(T-t)} - S = S[N(d_1) - 1] - Ke^{-r(T-t)}[N(d_2) - 1],$$ (A)

$$= Ke^{-r(T-t)}N(-d_2) - SN(-d_1),$$

where

$$d_1 = \frac{\log(S/K) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$$

and $d_2$ is given above. Calculating the vega using (A), we have

$$\text{Vega} = \frac{\partial V_p}{\partial \sigma} = SN'(d_1) \frac{\partial d_1}{\partial \sigma} - Ke^{-r(T-t)}N'(d_2) \frac{\partial d_2}{\partial \sigma}.$$
But using the calculation of ∆ from class, we have that the two coefficients of the partial derivatives are equal. Therefore, we have

\[ \text{Vega} = SN'(d_1) \left( \frac{\partial d_1}{\partial \sigma} - \frac{\partial d_2}{\partial \sigma} \right) = Sn(d_1) \frac{\partial}{\partial \sigma} \left( \frac{\sigma^2(T-t)}{\sigma \sqrt{T-t}} \right) = Sn(d_1) \sqrt{T-t}, \]

just as for the call option. If we buy a put and sell a call, we create a portfolio which is independent of \( \sigma \), and hence has vega 0. Therefore, the difference of the vegas of the put and call must also be equal to zero, and hence the vegas of the put and the call are equal.

3. The payoff diagram shown is for a condor spread. Denote its option value by \( V_z \).

(a) (6 points) Construct a portfolio \( \Pi_B \) of two bullish spreads \( B(t; K_b, K_s) \) that replicates this payoff. Draw the payoff diagrams of the individual spreads. Write \( V_z \) in terms of the values of various call options (you needn’t write out the full Black-Scholes formulas).

**Solution.** See the figure on the next page. The dotted line represents buying a bullish call spread \( B(t; K_-, K-z) \), and the dashed line represents selling a bullish call spread \( B(t; \bar{K} + z, K_+) \). Therefore, using our definition of the bullish call spread, we have that

\[
\Pi_B(S,t) = B(t; K_-, K-z) - B(t; \bar{K} + z, K_+)
\]

\[
V_z(S,t) = C(S,t; K-) - C(S,t; \bar{K} - z) - [C(S,t; \bar{K} + z) - C(S,t; K_+)]. \tag{B}
\]

(b) (9 points) Show that the sensitivity of \( V_z \) to changes in \( z \) is given by

\[
e^{-r(T-t)}[N(d_2^+) - N(d_2^-)], \quad d_2^+ = \frac{\log(S/(\bar{K} \pm z)) + (r - \sigma^2/2)(T-t)}{\sigma \sqrt{T-t}}.
\]

**Solution.** Taking the derivative of (B) with respect to \( z \), we have

\[
\frac{\partial V_z}{\partial z} = - \left[ - \frac{\partial C}{\partial K} (S,t; \bar{K} - z) \right] - \frac{\partial C}{\partial K} (S,t; \bar{K} + z),
\]
Decomposition of condor into two call spreads.

since the other two terms do not depend on $z$. Calculating the derivative with respect to strike, we have

$$
\frac{\partial C}{\partial K} = \frac{\partial}{\partial K} \left[ SN(d_1) - Ke^{-r(T-t)}N(d_2) \right] = Sn(d_1) \frac{\partial d_1}{\partial K} - e^{-r(T-t)} \left[ N(d_2) + Kn(d_2) \frac{\partial d_2}{\partial K} \right].
$$

Continuing to simplify, we have

$$
\frac{\partial C}{\partial K} = \frac{\partial d_1}{\partial K} \left[ Sn(d_1) - Ke^{-r(T-t)}n(d_2) \right] - e^{-r(T-t)}N(d_2) = -e^{-r(T-t)}N(d_2),
$$

where we have used the fact that $\partial d_1/\partial K = \partial d_2/\partial K$, and notes in class that the bracketed quantity is 0. Thus we have

$$
\frac{\partial V_z}{\partial z} = -e^{-r(T-t)}N(d_2^-) + e^{-r(t-T)}N(d_2^+) = e^{-r(T-t)}[N(d_2^+) - N(d_2^-)],
$$

$$
d_2^{\pm} = \log(S/\bar{K} \pm z) + (r - \sigma^2/2)(T-t) \frac{\sigma \sqrt{T-t}}{2},
$$

as required.

(c) (2 points) Use your answer to (b) to show that a condor is always cheaper than a butterfly for the same values of $K_\pm$.

Solution. Since $z > 0$ for a condor, we have that $d_2^+ < d_2^-$, so $N(d_2^+) < N(d_2^-)$ and $\partial V_z/\partial z < 0$. Hence $V_z$ is a strictly decreasing function of $z$. Hence a condor’s value
decreases with increasing $z$, and is always cheaper than a butterfly, which corresponds to $z = 0$.

(d) (2 points) Explain your answer to (d) financially.

*Solution.* A condor caps the payout in $[\bar{K} - z, \bar{K} + z]$ at a value less than the full value of the butterfly. Hence it should cost less than the butterfly.

4. (4 points) What is the put-call parity for options on an asset that pays a constant continuous dividend yield $\delta$?

*Solution.* Consider the portfolio given in class, namely

$$\Pi(S, t) = V_{\text{put}} - V_{\text{call}} + S.$$  

Because the stock pays dividends, at time $T$ we have

$$\Pi(S, T) = K - S + Se^{\delta(T-t)} \neq K.$$  

Therefore, we have too much stock to get a guaranteed payout. Instead, we need only $e^{-\delta(T-t)}$ shares of stock. In that case, we would have

$$\Pi(S, t) = V_{\text{put}} - V_{\text{call}} + Se^{-\delta(T-t)}$$

$$\Pi(S, T) = K - S + [Se^{-\delta(T-t)}]e^{\delta(T-t)} = K.$$  

Therefore, we have

$$V_{\text{put}} - V_{\text{call}} + e^{-\delta(T-t)}S = Ke^{-r(T-t)}.$$  

Note this is exactly the result we would obtain using the financial substitution $s = Se^{-\delta(T-t)}$ from class.