Homework Set 4 Solutions

1. (9 points) Let $C(S, t; T)$ be the value of a call option with exercise date $T$. Consider two otherwise identical calls with exercise dates $T_1, T_2$, $T_1 < T_2$. We wish to prove

$$C(S, t; T_1) \leq C(S, t; T_2), \quad 0 \leq t \leq T_1.$$  \hfill (A)

Use arbitrage arguments to prove the result financially.

Solution. There are many ways to do this, but for simplicity we consider the following portfolio:

$$\Pi(S, t) = C(S, t; T_1) - C(S, t; T_2).$$

and suppose that $C(S, t; T_1) > C(S, t; T_2)$ at some time $t \leq T_1$. At that point, sell $\Pi$ and invest the proceeds. Hence $\Pi(S, T_2) > 0$. Suppose that we instead hold the portfolio. If the first call option is not exercised, we have

$$\Pi(S, T_1) = -C(S, t; T_2)$$

$$\Pi(S, T_2) = -(S - K)^+ \leq 0.$$

If the first call option is exercised, we assume that we pay the strike and keep the stock to use later if needed. Hence we have

$$\Pi(S, T_1) = (S - K) - C(S, t; T_2)$$

$$\Pi(S, T_2) = S - Ke^{r(T_2 - T_1)} - (S - K)^+ = \begin{cases} S - Ke^{r(T_2 - T_1)} < 0, & S < K, \\ K - Ke^{r(T_2 - T_1)} < 0, & S > K. \end{cases}$$

Since $\Pi(S, T_2) \leq 0$, you are always guaranteed to make money by selling $\Pi$. Hence the supposition is false, and (A) is true.

2. Consider a portfolio $\Pi(S, t)$ that is short a call with strike $K_2$ and long a call on the same asset with strike $K_1 > K_2$.

(a) (6 points) Sketch the payoff function $\Pi(S, T)$, and write an algebraic expression for it.

Solution. The long call pays $(S - K_1)^+$, while the short call costs $-(S - K_2)^+$. Adding these two together, we have

$$\Pi(S, T) = \begin{cases} 0, & S < K_2, \\ S - K_2, & K_2 \leq S \leq K_1, \\ K_2 - K_1, & S > K_2. \end{cases}$$

(This is an example of a spread option, which we shall discuss later in class.) A diagram is shown above.

(b) (4 points) Explain financially why someone would ever buy this set of options.

*Solution.* The call with a lower strike should be worth more than the one with the higher strike, so selling the first and buying the second will leave you with a net profit. As long as that profit is not eaten up by the payoff at time $T$, you will make money. So this is a bet that the stock price will not exceed $K_2$ by too much.

(c) (9 points) Explain financially why $(K_2 - K_1)e^{-r(T-t)} \leq \Pi(S,t) \leq 0$ for all $0 \leq t \leq T$. Do **NOT** use the mathematical formulas for the calls.

*Solution.* Suppose 

$$\Pi(S,t) > 0. \quad (B.1)$$

Then sell the portfolio and invest the proceeds. At time $T$, we would then have $\Pi(S,t)e^{r(T-t)}$, which is greater than 0 by (B.1). But $\Pi(S,T) \leq 0$, so there would be an arbitrage opportunity: you could always make more money by selling the portfolio early rather than holding it until $T$. Similarly, suppose 

$$\Pi(S,t) < (K_2 - K_1)e^{-r(T-t)}. \quad (B.2)$$

Then someone else holding the contract would pay you to take it off their hands, so you would have the portfolio, plus $-\Pi(S,t)$ in cash. So at time $T$, you would have 

$$\Pi(S,T) - \Pi(S,t)e^{r(T-t)} > \Pi(S,T) + (K_1 - K_2),$$

where we have used (B.2). But this final sum is positive, so there is an arbitrage opportunity.
3. Consider an option with payoff function $\Lambda(S)$ and value $V(S, t)$, and let $C(S, t; K)$ be the value of a European call with strike price $K$. We wish to value the option as a portfolio of (an infinite number of) European calls with different strike prices:

$$V(S, t) = \int_0^\infty f(K)C(S, t; K) \, dK,$$

but we need to find the call density $f(K)$.

(a) (8 points) Use (4.1) to show that $\Lambda(0) = \Lambda'(0) = 0$. Interpret these conditions financially by considering the individual options in the portfolio.

\textbf{Solution.} Evaluating (4.1) at (payoff) time $T$, we have

$$V(S, T) = \Lambda(S) = \int_0^\infty f(K)C(S, T; K) \, dK = \int_0^\infty f(K)(S - K)^+ \, dK$$

$$= \int_0^S f(K)(S - K) \, dK$$

$$= S \int_0^S f(K) \, dK - \int_0^S Kf(K) \, dK,$$

where in the second line we have used the definition of the $(\cdot)^+$ notation. Evaluating (C) at $S = 0$, we have

$$\Lambda(0) = 0 - \int_0^0 Kf(K) \, dK = 0.$$

Taking the derivative of (B) using the Product Rule and the Fundamental Theorem of Calculus, we have

$$\Lambda'(S) = \int_0^S f(K) \, dK + S f(S) - S f(S) = \int_0^S f(K) \, dK,$$

$$\Lambda'(0) = \int_0^0 f(K) \, dK = 0.$$

We note that for any European call with $K > 0$, the option is worthless when $S = 0$, and moreover the derivative of its payoff is 0 at $S = 0$. Hence when assembling a portfolio of these options, it follows that the payoff of the portfolio and its derivative must also be 0 at $S = 0$.

(b) (4 points) Show that $f = \Lambda''$.

\textbf{Solution.} Taking the derivative of (D) using the Fundamental Theorem of Calculus, we have

$$\Lambda''(S) = f(S),$$

as desired.