Homework Set 1 Solutions

1. (9 points) Another common future is a currency future, where you enter into a contract (paid for in currency 1) to obtain one unit of currency 2 at some point $T$ in the future. Suppose that the exchange rate today between two currencies is $S(0)$ units of currency 1 per unit of currency 2. Also, assume that the risk-free rate for currency $j$ is $r_j$. Show that the currency forward price is given by

$$F(0; T) = S(0)e^{(r_1 - r_2)T}. \quad (1.1)$$

Make sure to explain your reasoning in terms of both the buyer and the seller.

Solution. The reasoning is equivalent to the case of dividends. The seller of the contract must have one unit of currency 2 at $t = T$. The seller wants to borrow in currency 1 now and buy currency 2, which they can then invest at $r_2$. Hence today the seller needs only $e^{-r_2T}$ of currency 2. Thus the seller borrows $S(0)e^{-r_2T}$ of currency 1 (which is equivalent to the desired amount, because of the exchange rate). At time $t = T$, the seller will then owe the amount on the right-hand side of (1.1), so $F(0; T)$ must be greater than that. Similarly, the buyer could borrow $e^{-r_2T}$ of currency 2 today from a third party, convert it to currency 1 [getting $S(0)e^{-r_2T}$] and invest it at $r_1$. At time $t = T$, the buyer would have the amount on the right-hand side of (1.1), but what would the buyer owe? The third party would want $e^{-r_2T}$, plus all the interest that the third party didn’t get in $[0, T]$, so the buyer would owe one full unit of currency 2. So $F(0; T)$ must be less than the right-hand side, so the proof is complete.

2. A prepaid forward contract is one where the buyer has to pay the seller at the time the contract is written, rather than at the settlement date.

(a) (4 points) Calculate the price $F(0; 0)$ of a prepaid forward contract in the case of no dividends. Make sure to explain your reasoning in terms of both the buyer and the seller.

Solution. The seller of the contract receives $F(0; 0)$ now and must buy the stock, whose value is $S(0)$. Hence $F(0; 0) \geq S(0)$. The buyer of the contract borrows the stock and sells it for $S(0)$, but then pays $F(0; 0)$. So $S(0) \geq F(0; 0)$. Hence $F(0; 0) = S(0)$. Hence buying a prepaid forward contract on a stock is equivalent to buying it now.

(b) (9 points) Suppose that a stock pays a fixed cash dividend amount $D(t_i)$ at fixed times $t_i$, where $0 < t_1 < t_2 < \cdots < t_n < T$. Show that in this case, the prepaid forward contract price is given by

$$F(0; 0) = S(0) - \sum_{i=1}^{n} D(t_i)e^{-r t_i}. \quad (1.2)$$
Verify that it reduces to your answer to (a) if \( D(t_i) = 0 \).

**Solution.** The seller of the contract receives \( F(0; 0) \) now and buys the stock, investing any leftovers at \( r \). As the seller receives the dividend payments, they are invested at \( r \). Hence, at time \( t = T \), the seller has the stock and all the dividend payments, but owes the stock. Therefore, at time \( T \), we have

\[
[F(0; 0) - S(0)]e^{rT} + \sum_{i=1}^{n} D(t_i)e^{r(T-t_i)} \geq 0,
\]

or the seller will lose money. Rewriting, we have

\[
F(0; 0) \geq S(0) - \sum_{i=1}^{n} D(t_i)e^{-rt_i}.
\]

The buyer of the contract borrows the share now, sells it for \( S(0) \), and buys the contract. Any remaining proceeds are invested at \( r \). The person lending the share is also going to want the income from the missing dividends as well. Therefore, we have

\[
[S(0) - F(0; 0)]e^{rT} + \sum_{i=1}^{n} D(t_i)e^{r(T-t_i)} \geq 0,
\]

or the buyer will lose money. Rewriting, we have

\[
F(0; 0) \leq S(0) - \sum_{i=1}^{n} D(t_i)e^{-rt_i},
\]

from which we have that the inequalities are an equality and

\[
F(0; 0) = S(0) - \sum_{i=1}^{n} D(t_i)e^{-rt_i},
\]

as required. Moreover, it reduces to the answer to (a) if \( D(t_i) = 0 \).

3. Let’s generalize our gambling example. Suppose that I have a $100 ticket that will pay a profit of \( P_1 \) if team 1 wins. Moreover, suppose that the current odds for team 2 winning are \( P_2:1 \).

(a) (5 points) Find the amount \( x \) I should bet on team 2 to hedge my bet.

**Solution.** If \( p \) is the probability that team 1 wins, the expected profits are as follows:

\[
\text{bet 1} = P_1p - 100(1 - p) \\
\text{bet 2} = -xp + P_2x(1 - p) \\
\text{both bets} = p[P_1 + 100 - x(P_2 + 1)] + P_2x - 100.
\]
Note here that the profit will be negative (in the amount of the bet) if the bet is lost. Therefore, to eliminate $p$ from the equation, I should bet

$$x = \frac{P_1 + 100}{P_2 + 1}.$$  

(b) (7 points) Find my expected profit $E$ in this case. Show that $E$ is an increasing function of $P_2$, bounded between the winning and losing sides of the $100$ bet. Interpret the limits that $P_2 \to 0$ and $P_2 \to \infty$.

Solution. Substituting the expression for $x$ into (A), we have

$$E = P_2 \frac{P_1 + 100}{P_2 + 1} - 100 = \frac{P_1 P_2 - 100}{P_2 + 1} \quad (C)$$

$$\frac{dE}{dP_2} = \frac{P_1}{P_2 + 1} - \frac{P_1 P_2 - 100}{(P_2 + 1)^2} = \frac{P_1 + 100}{(P_2 + 1)^2} > 0.$$

$$E(0) = -100$$

$$E(\infty) = P_1.$$  

In the limit that $P_2 \to 0$, team 1 is so unlikely to win that even though you have hedged your bet, your expected profit is the loss from the first bet. In the limit that $P_2 \to \infty$, team 1 is so likely to win that even though you have hedged your bet, your expected payoff is the gain from the first bet.

(c) (4 points) Under what conditions will your expected profit will be 0? Explain your answer in terms of your bet $x$.

Solution. Setting (C) equal to 0, we have $P_1 P_2 - 100 = 0$, or $P_1 = 100/P_2$. Substituting this result into (B), we have

$$x = \frac{100 + 100/P_2}{P_2 + 1} = 100 \frac{(P_2 + 1)/P_2}{P_2 + 1} = \frac{100}{P_2}.$$  

Note that in this case if I bet $x$ and win, I’ll get $100$, which is exactly the amount I lose from the first bet. Similarly, if I bet $x$ and lose, I lose $100/P_2 = P_1$, which is exactly the amount I win from the first bet.

(d) (2 points) Use your formulas to verify the example given in class.

Solution. In class, $P_1 = 50,000$ and $P_2 = 1/3$ (since the odds are 1:3). Substituting these values into our expression for $x$ and $E$, we have

$$x = \frac{50100}{4/3} = 37575, \quad E = \frac{(50000)(1/3) - 100}{4/3} = \frac{49700}{4} = 12425,$$

as derived in class.