Updates

1. The midterm will be distributed at the end of class on Tuesday, Apr. 13, and will be due back by 2 pm on Wednesday, Apr. 14.
2. The exam will cover up through Asian options, so you should complete the first three homework problems beforehand for practice.
3. You will need a small blue book for the exam.

Homework Set 6 (Revised)


Dividends

1. Consider a call option $C_n$ (expiration date $T$) on an asset that pays dividends at a constant percentage $\delta_n$ every $T/n$ before expiration. (In other words, $t_i = iT/n$, $i = 1, 2, \ldots, n - 1$.)

(a) (6 points) Show that

$$C_n(S,t) = C((1 - \delta_n)^kS,t;K), \quad t_{n-k-1} < t < t_{n-k}. \quad (6.1)$$

Now let $\delta_n = \delta T/n$, and consider the case where $n$ gets large.

(b) (3 points) What happens to $\delta_n$ in this case? What about the intervals listed in (6.1)? Use your answer to write an approximation for $k$ in terms of the other variables in the problem.

(c) (5 points) Use your answers to (b) to show that in the limit that $n \to \infty$, (6.1) reduces to the continuous case presented in class. Just talk about the variable substitutions; don’t go into the details of deriving the explicit solution.

Exotic Options

2. Let $C(t;K_2,T_2)$ be the value of a call option at time $t$ with strike $K_2$ and expiration $T_2$. Let $C_C(t;K_1,T_1)$ be a compound call on the underlying call option (call-on-call) with strike $K_1$ and expiration $T_1 < T_2$, and let $P_C(t;K_1,T_1)$ be a compound put on the underlying call option (put-on-call), also with strike $K_1$ and expiration $T_1$.

(a) (2 points) What is the payoff function for $P_C$?

(b) (3 points) Construct a put-call parity relationship for $C_C$ and $P_C$. 
Asian Options

3. Suppose that we want to introduce some exponential weighting into our average. This motivates a new definition of $I$:

$$I = \lambda \int_0^t e^{-\lambda(t-\tau)} S(\tau) \, d\tau.$$  

Note that this is not consistent with the definition given in class, since the integrand depends on $t$.

(a) (3 points) Show that $dI = f(S, I) \, dt$, and find $f$.
(b) (3 points) Derive a PDE for $V(S, I, t)$ similar to the one we gave in class.

Lookback Options

4. (7 points) Now consider an option which depends on the minimum $m$ of the asset price. As in class, define an appropriate variable $m_n$ which depends on the integral of $S$, and show that $\lim_{n \to \infty} m_n = m$.

American Options

5. Consider a perpetual American put option, which is one that can be exercised at any time.

(a) (3 points) Explain why the value of such an option is independent of time, and show that the Black-Scholes equation reduces to an Euler equation in this case.
(b) (6 points) Show that for any exercise boundary $S_e$, we have

$$V(S) = (K - S_e) \left( \frac{S}{S_e} \right)^{-2r/\sigma^2}. \quad (6.2)$$  

(Hint: What are the boundary conditions?)

The exact form for $S_e$ is beyond the scope of this homework.