Homework Set 3 (Revised)

Read sections 2.3 and 3.5.

Itô’s Lemma

1. Consider the lognormal distribution for $S$ given in class:

   $$p(s) = \frac{1}{\sigma S \sqrt{2\pi t}} \exp \left( -\frac{[\log s - (\mu - \sigma^2/2)t]^2}{2\sigma^2 t} \right), \quad s > 0.$$  
   \hspace{1cm} (3.1)

   If you use a computer to help with these integrals, provide a printout.

   (a) (5 points) Show by direct calculation that $\langle \log S \rangle = (\mu - \sigma^2/2)t$.
   (b) (9 points) Calculate $\langle S \rangle$. Compare $\log \langle S \rangle$ to $\langle \log S \rangle$.

The Black-Scholes Equation

2. (7 points) page 57, number 8. Here $S$ is the value of the asset at time $t$. You should derive the result from financial first principles.
3. Suppose that an option $V$ depends on $n$ assets which satisfy the following stochastic differential equation:

$$dS_i = \sigma_i S_i dX_i + \mu_i S_i \, dt, \quad i = 1, 2, \ldots, n.$$  \hfill (3.2a)

Here the Wiener processes $dX_i$ satisfy

$$E[dX_i] = 0, \quad E[dX_i^2] = dt$$

as usual, but the asset price changes are correlated with

$$E[dX_i dX_j] = \rho_{ij} \, dt, \quad -1 \leq \rho_{ij} = \rho_{ji} \leq 1.$$  

In that case, it can be shown using Itô’s Lemma that

$$dV = \left( \frac{\partial V}{\partial t} + \sum_{i=1}^{n} \mu_i S_i \frac{\partial V}{\partial S_i} + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \rho_{ij} \sigma_i \sigma_j S_i S_j \frac{\partial^2 V}{\partial S_i \partial S_j} \right) dt$$

$$+ \sum_{i=1}^{n} \frac{\partial V}{\partial S_i} \sigma_i S_i dX_i,$$ \hfill (3.2b)

where we note that $\rho_{ii} = 1$ from the definition of $E[X_i^2].$

(a) (12 points) Find the values $\Delta_i$ such that the changes in the portfolio

$$\Pi = \sum_{i=1}^{n} \Delta_i S_i - V$$

have no stochastic component. Verify your answer matches notes in class in the case that $n = 1.$

(b) (7 points) Find the Black-Scholes equation for $V$ in this case. Again verify your answer matches notes in class in the case that $n = 1.$