Homework Set 2

Read sections 2.1 and 2.2.

The Random Walk

1. Consider the ruin problem. Suppose that a gambler starts with wealth \( w \), and plays a game where in each round he wins 1 with probability \( p \), and loses 1 with probability \( 1 - p \). The game is assumed to last until the gambler loses all his money.

   (a) (5 points) Let \( P(w) \) be the probability that the gambler loses all his money, given a wealth of \( w \). Explain why

   \[
   P(w) = pP(w + 1) + (1 - p)P(w - 1),
   \]  

   and calculate the boundary conditions \( P(0) \) and \( P(\infty) \).

   Equations of the form (2.1) are called second-order constant-coefficient homogeneous linear difference equations; they have solutions of the form

   \[
   P(w) = c_1 \lambda_1^w + c_2 \lambda_2^w
   \]  

   for some constants \( c_j \), \( \lambda_j \). (Note the similarity to the solution for an ODE.)

   (b) (4 points) Find the \( \lambda_j \) that satisfy (2.1).

   (c) (9 points) Using your answer to (b) and the boundary conditions you derived in (a), show that

   \[
   P(w) = \begin{cases} 
   1, & p \leq \frac{1}{2}, \\
   \left(\frac{1 - p}{p}\right)^w, & p > \frac{1}{2}.
   \end{cases}
   \]

   Interpret your results in the gambling context.
Continuous Limit

2. We now derive our probability density function using Laplace transform techniques. Consider the equation

$$\frac{\partial u}{\partial t} = -r \frac{\partial u}{\partial x} + \frac{\sigma^2}{2} \frac{\partial^2 u}{\partial x^2}, \quad u(x, 0) = \delta(x).$$

It can be shown that if we let

$$z = x - rt, \quad u(x, t) = v(z, t),$$

the resulting equation for $v$ is given by

$$\frac{\partial v}{\partial t} = \frac{\sigma^2}{2} \frac{\partial^2 v}{\partial z^2}, \quad v(z, 0) = \delta(z). \tag{2.3}$$

(a) (13 points) Take the Laplace transform of (2.3) to obtain the following system:

$$\frac{\sigma^2}{2} \frac{d^2 \hat{v}}{dz^2} - s \hat{v} = 0, \quad z \neq 0, \tag{2.4a}$$

$$\hat{v}(0^+) = \hat{v}(0^-), \quad \frac{d\hat{v}}{dz}(0^+) - \frac{d\hat{v}}{dz}(0^-) = -\frac{2}{\sigma^2}, \tag{2.4b}$$

where $\hat{v}$ is the Laplace transform of $v$, and $s$ is the Laplace transform variable. (Hint: Integrate your transformed equation from $z = 0^-$ to $z = 0^+$. ) What are reasonable conditions on $\hat{v}$ as $z \rightarrow \pm \infty$?

(b) (9 points) Solve (2.4) and use tables to determine that

$$v(z, t) = \frac{1}{\sigma \sqrt{2\pi t}} \exp \left( -\frac{z^2}{2\sigma^2 t} \right),$$

which matches the results in class.