MATH 617-010
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Introduction to Applied Mathematics II
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Review of Frobenius Series

Consider the second-order ODE

$$y'' + p(z)y' + q(z)y = 0, \tag{1}$$

and suppose we want to obtain a solution about $z = 0$.

Ordinary Point

If p and q are both analytic at $z = 0$, we call $z = 0$ an *ordinary point* of the equation and we may obtain two linearly independent solutions by substituting in the Taylor series

$$y(z) = \sum_{n=0}^{\infty} a_n z^n$$

and equating coefficients of z^n. Here a_0 and a_1 will be arbitrary (to yield two linearly independent solutions), but all the other a_n can be expressed as functions of a_0 and a_1.

Regular Singular Point

Suppose that p and q are not both analytic at $z = 0$, but zp and $z^2 q$ are:

$$zp = \sum_{n=0}^{\infty} p_n z^n, \qquad z^2 q = \sum_{n=0}^{\infty} q_n z^n. \tag{2}$$

Then we call $z = 0$ a *regular singular point* of the equation. The simplest case is an Euler equation

$$y'' + \frac{p_0}{z} y' + \frac{q_0}{z^2} y = 0.$$

In this case, we may obtain two linearly independent solutions by substituting in the modified Taylor series

$$y(z) = z^c \sum_{n=0}^{\infty} a_n z^n \tag{3}$$

and equating coefficients of z^{n+c}. Here only a_0 will be arbitrary; the two linearly independent solutions arise since c can take on only one of two values $\{c_1, c_2\}$, which are roots of the *indicial equation*

$$c(c-1) + p_0 c + q_0 = 0.$$

This quadratic equation is the coefficient of z^c once one substitutes (3) into (1). Since only a_0 is arbitrary, we see that all the other a_n can be expressed as functions of a_0.

Clearly if the quadratic has only one root, then we must find a second linearly independent solution in some other way. In addition, it can be shown that if $c_1 - c_2$ is an integer, the two solutions given by (3) will also not be linearly independent.

For more details, a good place to start is the book *Elementary Differential Equations and Boundary Value Problems* by Boyce and DiPrima, which we use on campus in our undergraduate courses.