

Envelopes

For the problem

$$\frac{\partial \rho}{\partial t} + \rho \frac{\partial \rho}{\partial x} = 0, \quad \rho(x, 0) = -\tanh x,$$

the characteristics are given by

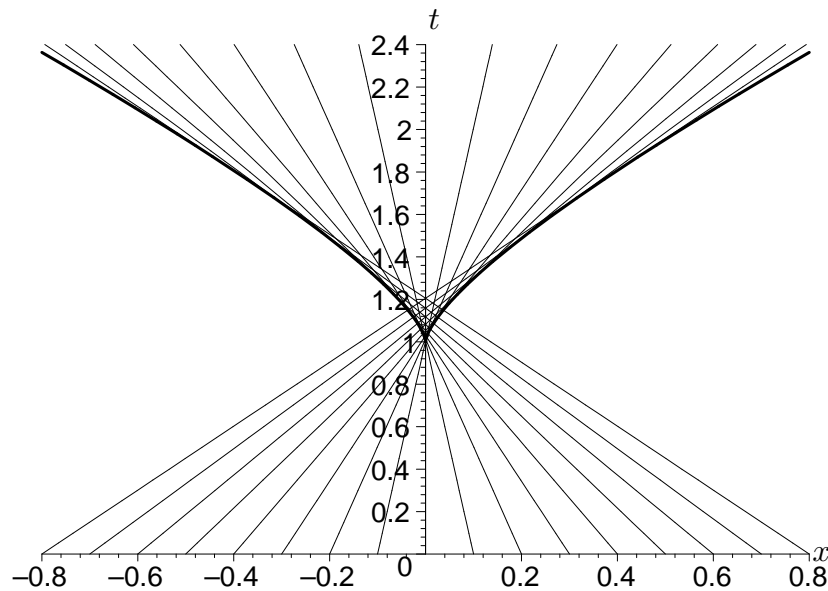
$$x = \xi - (\tanh \xi)t, \tag{1}$$

and the envelope of intersecting characteristics is given by (1) and

$$0 = 1 - (\operatorname{sech}^2 \xi)t. \tag{2}$$

We also determined that $t_B = 1$.

Below are plotted characteristics given by (1) for $\xi = -0.8$ through $\xi = 0.8$ incremented by 0.1, as well as the envelope (thick line). Note that the envelope has a cusp at t_B .



Characteristics given by (1) (thin lines) and envelope given by (1) and (2) (thick line).

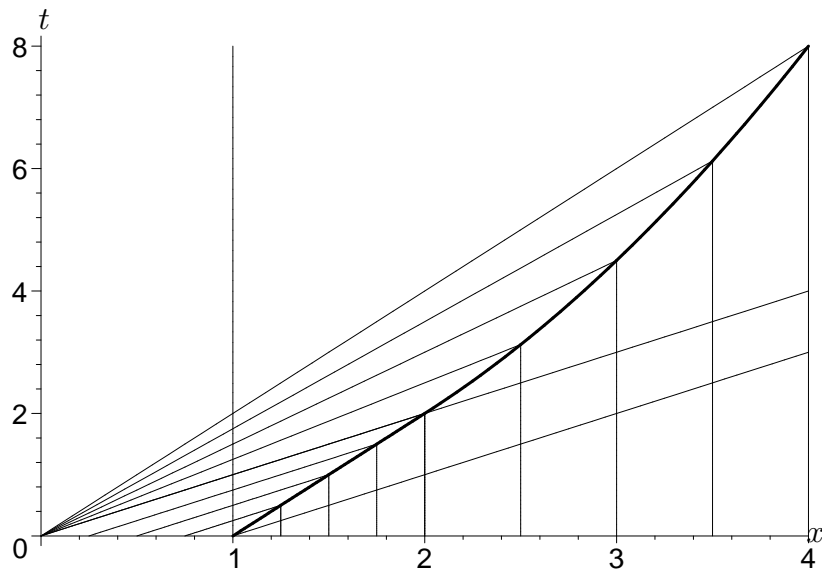
Shock Structure

For the problem

$$\frac{\partial \rho}{\partial t} + \rho \frac{\partial \rho}{\partial x} = 0, \quad \rho(x, 0) = \begin{cases} 1, & x \in [0, 1], \\ 0, & \text{else,} \end{cases} \quad (3)$$

we determined that the shock position is given by

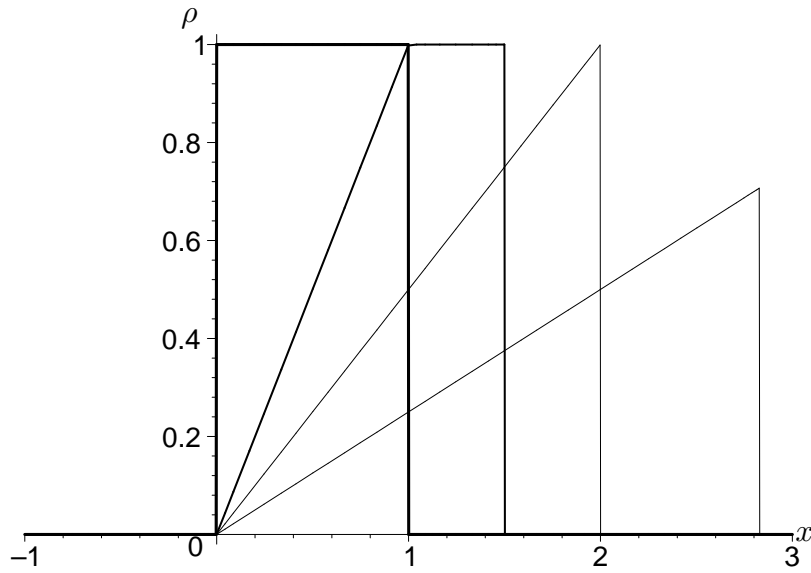
$$s(t) = \begin{cases} 1 + t/2, & t \in [0, 2], \\ \sqrt{2t}, & t \geq 2. \end{cases} \quad (4)$$



Characteristics for (3) (thin lines) and shock position given by (4) (thick line).

The characteristic diagram is plotted above. Note that the envelope of crossings is given by the lines $x = t + 1$ and $x = 1$. The shock position is indicated by the thick line. Note that it changes from a straight line for $x \in [1, 2]$ to a parabola for $x \geq 2$.

Shock Profiles



Profiles for $t = 0, 1, 2,$ and 4 (in decreasing order of thickness).

Profiles for ρ are shown above. Note that for $t = 4$ (once the fan characteristics determine the front position), the overall height of the sawtooth declines.