Homework Set 5 Solutions

1. Consider the following ordinary differential equation:

\[ y'' + 9y = x + \alpha, \quad y(0) = y(\pi) = 0. \]  \hspace{1cm} (5.1)

(a) (4 points) Find all \( \alpha \) for which (5.1) has a solution.

Solution. We note that \( \phi = \sin 3x \) is a solution to the homogeneous problem. From
the Fredholm alternative, we see that for a solution to exist we must have

\[
\langle x + \alpha, \sin 3x \rangle = \int_0^\pi x \sin 3x + \alpha \sin 3x\, dx = 0
\]

\[
\left[ -\frac{x + \alpha}{3} \cos 3x + \frac{\sin 3x}{9} \right]_0^\pi = \frac{\pi + \alpha}{3} + \frac{\alpha}{3} = 0
\]

\[ \alpha = -\frac{\pi}{2}. \]

(b) (6 points) Write the appropriate Fourier series solution for those special values
of \( \alpha \). Is the solution unique?

Solution. We expand our functions in series of eigenfunctions:

\[
y(x) = \sum_{n=0}^{\infty} y_n \sin nx, \quad f(x) = \sum_{n=0}^{\infty} f_n \sin nx
\]

(A)

The boundary conditions are homogeneous, so the convergence is uniform. Hence we may
substitute (A) into (5.1) and equate term-by-term:

\[
\sum_{n=1}^{\infty} -n^2 y_n \sin nx + 9y_n \sin nx = \sum_{n=1}^{\infty} f_n \sin nx,
\]

\[
(9 - n^2)y_n = f_n
\]

\[ y_n = \frac{f_n}{9 - n^2}. \]  \hspace{1cm} (B.1)

Therefore, as guaranteed by the Fredholm alternative, there will be a solution only if
\( f_3 = 0 \), in which case \( y_3 \) will be arbitrary. Calculating \( f_n \), we have

\[
f_n = \frac{2}{\pi} \int_0^\pi (x + \alpha) \sin nx\, dx = \frac{2}{\pi} \left[ -\frac{x - \pi/2}{n} \cos nx + \frac{\sin nx}{n^2} \right]_0^\pi = \frac{2}{n\pi} \left[ -\frac{\pi}{2} (-1)^n - \frac{\pi}{2} \right]
\]

\[ = -\frac{2}{n}, \quad n \text{ even}, \]  \hspace{1cm} (B.2)
where we have used the fact that $\alpha = -\pi/2$. Hence $f_3 = 0$, as required. Substituting (B) into (A), we have

$$y_{2m} = -\frac{2}{2m[9 - (2m)^2]}$$

$$y(x) = -\sum_{m=1}^{\infty} \frac{\sin 2mx}{m(9 - 4m^2)} + a_3 \sin 3x,$$

and hence the solution is not unique.

2. (11 points) Write the solution to

$$\frac{d}{dx} \left( (1 - x^2) \frac{du}{dx} \right) = f(x), \quad u \text{ bounded at } x = -1, x = 1,$$

in terms of a (regular or modified) Green’s function. Be sure to state any restrictions on $f$ and how they affect any arbitrariness in your solution.

**Solution.** There is an eigenfunction for the operator. In particular, any constant function will satisfy

$$\frac{d}{dx} \left( (1 - x^2) \frac{du}{dx} \right) = 0, \quad u \text{ bounded at } x = -1, x = 1,$$

and so we choose $u_0 = 1$ as our test eigenfunction. Here $\rho = 1$ and so the inner product is given by

$$\langle f, g \rangle = \int_{-1}^{1} fg \, dx \quad \Rightarrow \quad ||1||^2 = \int_{-1}^{1} dx = 2.$$

Therefore, we choose our normalized eigenfunction to be $\phi_0 = 1/\sqrt{2}$. Then we must solve the following problem for our modified Green’s function $G(x|\xi)$:

$$\frac{d}{dx} \left( (1 - x^2) \frac{dG}{dx} \right) = \delta(x - \xi) - \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}}, \quad G \text{ bounded at } x = -1, x = 1,$$

$$G(\xi^-) = G(\xi^+), \quad \left[ (1 - x^2) \frac{dG}{dx} \right]_{x=\xi} = 1.$$

Note the unusual jump condition because of the form of the equation.

Solving the homogeneous portion, we have

$$(1 - x^2) \frac{dG}{dx} = -\frac{x + A}{2}$$

$$\frac{dG}{dx} = -\frac{x + A}{2(1 - x^2)} = \frac{1 - A}{4(1 + x)} - \frac{1 + A}{4(1 - x)}$$

$$G = \frac{1 - A}{4} \log(x + 1) + \frac{1 + A}{4} \log(1 - x) + B.$$
To satisfy the boundedness conditions, we must set \( A = \pm 1 \), so we obtain

\[
G = \begin{cases} 
\frac{1}{2} \log(1 - x) + B_-, & -1 \leq x < \xi, \\
\frac{1}{2} \log(1 + x) + B_+, & \xi < x \leq 1.
\end{cases}
\]

To satisfy the continuity conditions, we must have that

\[
\frac{1}{2} \log(1 - \xi) + B_- = \frac{1}{2} \log(1 + \xi) + B_+
\]

\[
G = \begin{cases} 
\frac{1}{2} \log(1 - x)(1 + \xi) + B, & -1 \leq x < \xi, \\
\frac{1}{2} \log(1 + x)(1 - \xi) + B, & \xi < x \leq 1.
\end{cases}
\]

Checking the jump condition, we obtain

\[
\left[(1 - x^2) \frac{dG}{dx}\right]_{x=\xi} = (1 - \xi^2) \left\{ \frac{1}{2(1 + \xi)} - \left[ -\frac{1}{2(1 - \xi)} \right] \right\} = \frac{1 - \xi^2}{2} \left( \frac{2}{(1 + \xi)(1 - \xi)} \right) = 1,
\]

and so as expected we have satisfied the jump condition while still retaining the arbitrary constant \( B \), which is just a multiple of \( \phi_0 \).

By the Fredholm alternative, we know that \( \langle f, \phi_0 \rangle = 0 \) for a solution to exist, so we may take \( B = 0 \) without loss of generality to write our solution as

\[
\begin{align*}
u(x) &= \int_{-1}^{1} f(x)G(x|\xi) \, d\xi + C, \\
G(x|\xi) &= \begin{cases} 
\frac{1}{2} \log(1 - x)(1 + \xi), & -1 \leq x < \xi, \\
\frac{1}{2} \log(1 + x)(1 - \xi), & \xi < x \leq 1.
\end{cases}
\end{align*}
\]

Here the arbitrary constant \( C \) is a multiple of the eigenfunction to the homogeneous problem, and hence can be added on without loss of generality.

3. Consider the following problem:

\[
u'' + \lambda^2 u = 0, \quad x \in [0, 1]; \quad u(0) = 0, \quad u'(1) = \lambda u(1).
\]

(a) (5 points) Rewrite the problem as

\[
\mathcal{L}u = \begin{pmatrix} u_2' \\ -u_1' \end{pmatrix} = \lambda \mathbf{u} = \lambda \begin{pmatrix} u_1 \\ u_2 \end{pmatrix},
\]

where \( \mathbf{u} \) is independent of \( \lambda \). Define the components of \( \mathbf{u} \), and show that (5.2) and (5.3) are equivalent.
Solution. From (5.3), we see that
\[ u'_2 = \lambda u_1, \quad -u'_1 = \lambda u_2 \quad \implies \quad -u''_2 = \lambda^2 u_2. \]
Therefore, \( u_2 = \alpha u \), since any multiple of \( u \) also satisfies (5.2). We also note from (5.3) that
\[ u_1 = \frac{u'_2}{\lambda} = \frac{u(1)u'_2}{u'(1)}, \]
where we have used (5.2). Therefore, for simplicity we let
\[ u = \begin{pmatrix} u(1) u' \\ u'(1) u \end{pmatrix}, \quad \text{(C)} \]
so we have
\[ \mathcal{L}u - \lambda u = \begin{pmatrix} u'(1)u' \\ -u(1)u'' \end{pmatrix} - \lambda \begin{pmatrix} u(1)u' \\ u'(1)u \end{pmatrix} = \begin{pmatrix} [u'(1) - \lambda u(1)]u' \\ -u(1)(-\lambda^2 u) - \lambda[\lambda u(1)]u' \end{pmatrix} = 0, \]
where we retain the boundary condition that \( u(0) = 0 \).

(b) (5 points) Define an inner product and show that (5.3) is self-adjoint.

Solution. Following notes in class, we define
\[ \langle u, v \rangle = \int_0^1 u_1 v_1 \, dx + \int_0^1 u_2 v_2 \, dx = u(1)v(1) \int_0^1 u'v' \, dx + u'(1)v'(1) \int_0^1 uv \, dx, \quad \text{(D.1)} \]
where (D.1) is a general definition and (D.2) holds when \( u \) and \( v \) are defined as in (C). Checking self-adjointness, we have
\[ \langle \mathcal{L}u, v \rangle = v(1)u'(1) \left[ \int_0^1 u'v' \, dx - v'(1)u(1) \int_0^1 u''v \, dx \right] \]
\[ = v(1)u'(1) \left[ u(1)v'(1) - u(0)v'(0) - \int_0^1 uv \, dx \right] \]
\[ - v'(1)u(1) \left[ u'(1)v(1) - u'(0)v(0) - \int_0^1 u'v' \, dx \right] \]
\[ = v(1)u'(1) \int_0^1 u'v' \, dx - v(1)u'(1) \int_0^1 uv \, dx + v'(1)u(1)u'(0)v(0) \]
\[ = \langle u, \mathcal{L}v \rangle + v'(1)u(1)u'(0)v(0), \]
and hence the problem is self-adjoint once we require that \( v(0) = 0 \).

(c) (4 points) Solve for the eigenfunctions \( u_n \) and eigenvalues \( \lambda_n \) of (5.3).
Solution. We may find the eigenfunctions of (5.3) by using the solution of (5.2), which is given by \( u_n = \sin \lambda_n x \), where

\[
\lambda_n \cos \lambda_n = \lambda_n \sin \lambda_n
\]
\[
\tan \lambda_n = 1
\]
\[
\lambda_n = \left( n + \frac{1}{4} \right) \pi
\]
\[
u_n(1) = \sin \left( n + \frac{1}{4} \right) \pi = \frac{(-1)^n}{\sqrt{2}}
\]
\[
u_n = \left( \lambda_n u_n(1) \cos \lambda_n x \right) = \frac{(-1)^n}{\sqrt{2}} \left( n + \frac{1}{4} \right) \pi \left( \cos \left( n + \frac{1}{4} \right) \pi x \right).
\]

We have constructed our eigenfunctions directly from the definition. However, any multiple of an eigenfunction is also an eigenfunction, so the vector without the coefficient is also an acceptable answer.

(d) (5 points) Verify that

\[
\langle u_m, u_n \rangle = 0, \quad m \neq n,
\]

and calculate \( ||u_n||^2 \).

Solution. We begin with the case where \( m \neq n \). We use (D.1) as the vectors have already been defined:

\[
\langle u_m, u_n \rangle = \frac{(-1)^{m+n}}{2} \left( n + \frac{1}{4} \right) \left( m + \frac{1}{4} \right) \int_0^1 \cos \left[ \left( m + \frac{1}{4} \right) \pi x \right] \cos \left[ \left( n + \frac{1}{4} \right) \pi x \right] + 
\]
\[
\sin \left[ \left( m + \frac{1}{4} \right) \pi x \right] \sin \left[ \left( n + \frac{1}{4} \right) \pi x \right] \, dx
\]
\[
= \frac{(-1)^{m+n}}{2} \left( n + \frac{1}{4} \right) \left( m + \frac{1}{4} \right) \int_0^1 \cos \left( m - n \right) \pi x \, dx
\]
\[
= \frac{(-1)^{m+n}}{2 \left( m - n \right) \pi} \left[ \sin \left( m - n \right) \pi x \right]_0^1 = 0.
\]

Checking the case \( m = n \), we substitute that equality into (E) to obtain

\[
\langle u_n, u_n \rangle = \frac{(-1)^{n+n}}{2} \left( n + \frac{1}{4} \right) \int_0^1 \, dx = \frac{\pi^2}{2} \left( n + \frac{1}{4} \right) ^2.
\]