Homework Set 4 Solutions (Revised)

1. (8 points) Let $f(x)$ be defined for all $x$ and have simple zeroes at $x_1 < x_2 < \cdots < x_n$. Show that

$$\frac{d(\text{sgn}(f(x)))}{dx} = \sum_{j=1}^{n} 2\delta(x - x_j) \text{sgn}(f'(x_j)).$$

Solution. Let $x_0 = -\infty$, $x_{n+1} = \infty$. We note that $\text{sgn}(f(x)) = 1$ for $x \in (x_j, x_{j+1})$ only when $f(x)$ changes from negative to positive at $x_j$. This occurs when $f'(x_j) > 0$. Similarly, $\text{sgn}(f(x)) = -1$ for $x \in (x_j, x_{j+1})$ when $f'(x_j) < 0$. Thus we have that

$$\text{sgn}(f(x)) = \text{sgn}(f'(x_j)), \quad x \in (x_j, x_{j+1}). \tag{A}$$

Integrating our relationship against a test function, we have

$$\int_{-\infty}^{\infty} \frac{d(\text{sgn}(f(x)))}{dx} \phi(x) \, dx = \sum_{j=0}^{n} \int_{x_j}^{x_{j+1}} \frac{d(\text{sgn}(f(x)))}{dx} \phi(x) \, dx$$

$$= \sum_{j=0}^{n} [\text{sgn}(f(x))\phi(x)]_{x_j}^{x_{j+1}} - \int_{x_j}^{x_{j+1}} \text{sgn}(f(x))\phi'(x) \, dx$$

$$= [\text{sgn}(f(x))\phi(x)]_{x_0}^{x_{n+1}} - \sum_{j=0}^{n} \text{sgn}(f'(x_j)) \int_{x_j}^{x_{j+1}} \phi'(x) \, dx,$$

where we have used (A) and the alternating signs in the first evaluation to cancel the middle terms. But $\phi(\pm\infty) = 0$, we have

$$\int_{-\infty}^{\infty} \frac{d(\text{sgn}(f(x)))}{dx} \phi(x) \, dx = -\sum_{j=0}^{n} \text{sgn}(f'(x_j))[\phi(x_{j+1}) - \phi(x_j)]$$

$$= \sum_{j=0}^{n} \text{sgn}(f'(x_{j+1}))\phi(x_{j+1}) + \text{sgn}(f'(x_j))\phi(x_j),$$

where we have used the fact that the zeroes are simple, so the signs of the derivatives alternate. Continuing to simplify using the fact that $\phi(\pm\infty) = 0$, we have

$$\int_{-\infty}^{\infty} \frac{d(\text{sgn}(f(x)))}{dx} \phi(x) \, dx = 2 \sum_{j=1}^{n} \text{sgn}(f'(x_j))\phi(x_j)$$

$$= \int_{-\infty}^{\infty} \sum_{j=1}^{n} 2\delta(x - x_j) \text{sgn}(f'(x_j))\phi(x) \, dx,$$
as required.

2. Consider the following problem:

\[ G'' - k^2(x)G = \delta(x-\xi), \quad G \to 0 \text{ as } |x| \to \infty, \quad k(x) = \begin{cases} k_-, & x < 0, \\ k_+, & x > 0, \end{cases} \tag{4.1} \]

where the \( k_\pm \) are unequal positive constants.

(a) (11 points) Show that for \( \xi > 0 \),

\[ G = \begin{cases} \frac{1}{2k_+} \left( \frac{k_- - k_+}{k_+ + k_-} e^{-k_+(x+\xi)} - e^{-k_+|x-\xi|} \right), & x \geq 0, \\ -\frac{e^{k_-x-k_+\xi}}{k_+ + k_-}, & x \leq 0. \end{cases} \tag{4.2} \]

**Solution.** We begin by examining the case where \( \xi > 0 \). In that case, the solution in the region \( x > 0 \) becomes

\[ G = \begin{cases} A_+ e^{k_+x} + A_- e^{-k_+x}, & 0 \leq x \leq \xi, \\ A_+ e^{-k_+x}, & \xi \leq x, \end{cases} \]

where we have used the far-field condition. Then by continuity at \( x = \xi \) we have

\[ G = \begin{cases} \left( B_+ e^{k_+\xi} + B_- e^{-k_+\xi} \right) e^{-k_+\xi}, & 0 \leq x \leq \xi, \\ \left( B_+ e^{k_+\xi} + B_- e^{-k_+\xi} \right) e^{-k_+x}, & \xi \leq x. \end{cases} \]

The jump condition at \( x = \xi \) is trivially given by

\[ G'(\xi^+) - G'(\xi^-) = 1 \]

\[ -k_+(B_+ e^{k_+\xi} + B_- e^{-k_+\xi}) e^{-k_+\xi} - k_+(B_+ e^{k_+\xi} - B_- e^{-k_+\xi}) e^{-k_+\xi} = 1 \]

\[ -2k_+ B_+ = 1 \]

\[ G = \begin{cases} \left( B_- e^{-k_+x} - \frac{e^{k_+x}}{2k_+} \right) e^{-k_+\xi}, & 0 \leq x \leq \xi, \\ \left( B_- e^{-k_+\xi} - \frac{e^{k_+\xi}}{2k_+} \right) e^{-k_+x}, & \xi \leq x. \end{cases} \]

\[ G = B_- e^{-k_+(x+\xi)} - \frac{e^{-k_+|x-\xi|}}{2k_+}, \quad x \geq 0. \]

Now examining the region for \( x < 0 \), we see that

\[ G = A e^{k_-x}, \quad x < 0, \]
where we have satisfied the far-field condition. Continuity at \( x = 0 \) yields

\[
G = \begin{cases} 
B_- e^{-k_+(x+\xi)} - \frac{e^{-k_+|x-\xi|}}{2k_+}, & x \geq 0, \\
(B_- e^{-k_+\xi} - \frac{e^{-k_+|x|}}{2k_+}) e^{k_- x}, & x \leq 0,
\end{cases}
\]

where we have used the fact that \( \xi > 0 \). Continuity of the derivative at \( x = 0 \) yields

\[
-k_+ B_- e^{-k_+\xi} - k_+ \frac{e^{-k_+\xi}}{2k_+} = k_- \left( B_- e^{-k_+\xi} - \frac{e^{-k_+|x-\xi|}}{2k_+} \right)
\]

\[
-(k_+ + k_-) B_- = \frac{k_+ - k_-}{2k_+}
\]

\[
B_- = \frac{k_+ - k_-}{2k_+(k_+ + k_-)}
\]

\[
G = \begin{cases} 
\frac{k_+ - k_-}{2k_+(k_+ + k_-)} e^{-k_+(x+\xi)} - \frac{e^{-k_+|x-\xi|}}{2k_+}, & x \geq 0, \\
\left( \frac{k_+ - k_-}{2k_+(k_+ + k_-)} - \frac{1}{2k_+} \right) e^{k_- x-k_+\xi}, & x \leq 0,
\end{cases}
\]

\[
= \begin{cases} 
\frac{1}{2k_+} \left( k_- - k_+ e^{-k_+(x+\xi)} - e^{-k_+|x-\xi|} \right), & x \geq 0, \\
- \frac{e^{k_- x-k_+\xi}}{k_- + k_+}, & x \leq 0,
\end{cases}
\]

as required.

(b) (8 points) Find the analogous expression for \( \xi < 0 \) and show that they agree in the limit that \( \xi = 0 \).

**Solution.** We note that (4.1) is symmetric under the mapping

\[
\xi \mapsto -\xi, \quad x \mapsto -x, \quad k_- \mapsto k_+, \quad k_+ \mapsto k_-.
\]

Substituting this result into (4.2), we obtain

\[
G = \begin{cases} 
\frac{1}{2k_-} \left( k_- - k_+ e^{-k_-(x-\xi)} - e^{-k_-|x-\xi|} \right), & x \geq 0, \\
-e^{-k_+ x+k_-\xi}, & x \leq 0,
\end{cases}
\]

\[
= \begin{cases} 
\frac{1}{2k_-} \left( k_- - k_+ e^{-k_-(x+\xi)} - e^{-k_-|\xi-x|} \right), & x \leq 0, \\
- \frac{e^{-k_+ x+k_-\xi}}{k_- + k_+}, & x \geq 0.
\end{cases}
\]
In the limit that $\xi \to 0^-$, the above expression becomes

$$G = \begin{cases} 
\frac{1}{2k_-} \left( \frac{k_+ - k_-}{k_- + k_+} e^{k_- x} - e^{-k_- |x|} \right), & x \leq 0, \\
-\frac{e^{-k_+ x}}{k_- + k_+}, & x \geq 0,
\end{cases}$$

while in the limit that $\xi \to 0^+$, (4.2) becomes

$$G = \begin{cases} 
\frac{1}{2k_+} \left( \frac{k_- - k_+}{k_- + k_+} e^{-k_+ x} - e^{-k_+ |x|} \right), & x \geq 0, \\
-\frac{e^{k_- x}}{k_- + k_+}, & x \leq 0.
\end{cases}$$

which matches, once you realize that the line order has been permuted.

3. Consider the problem

$$y'' + k^2 y = f(x), \quad k > 0; \quad y'(0) = y'(\pi) = 0. \quad (4.3)$$

(a) (5 points) Solve (4.3) using an eigenfunction expansion. Are there any values of $k$ for which your formula breaks down?

Solution. Solving the eigenfunction problem, we have

$$u'' + \lambda^2 u = 0$$

$$u(x) = A \cos \lambda x + B \sin \lambda x$$

$$u'(0) = B \lambda = 0 \implies B = 0$$

$$u'(\pi) = -\lambda A \sin \lambda \pi = 0 \implies \lambda = n.$$ 

Therefore, motivated by the Fourier cosine series, we let

$$y(x) = \frac{y_0}{2} + \sum_{n=1}^{\infty} y_n \cos nx, \quad (B)$$
and similarly for \( f(x) \) with
\[
f_n = \frac{2}{\pi} \int_0^\pi f(x) \cos nx \, dx.
\]
Then (as in the class notes) the convergence is uniform, so substituting (B) into (4.3), we obtain
\[
\frac{k^2 y_0}{2} + \sum_{n=1}^{\infty} -n^2 y_n \cos nx + k^2 y_n \cos nx = \frac{f_0}{2} + \sum_{n=0}^{\infty} f_n \cos nx
\]
\[
y_n = \frac{f_n}{k^2 - n^2}.
\]
Hence the formula breaks down if \( k = n \).

(b) (8 points) Find the Green’s function for (4.3). Are there any values of \( k \) for which your formula breaks down?

\textit{Solution.} Solving the homogeneous problem, we have
\[
G'' + k^2 G = 0
\]
\[
G = A \sin kx + B \cos kx.
\]
Therefore, solving the boundary conditions, we have that
\[
G = \begin{cases} 
A \cos kx, & 0 \leq x \leq \xi, \\
B \cos k(\pi - x), & \xi \leq x \leq \pi.
\end{cases}
\]
Solving the continuity condition, we obtain
\[
G = \begin{cases} 
C \cos kx \cos k(\pi - x), & 0 \leq x \leq \xi, \\
C \cos k\xi \cos k(\pi - x), & \xi \leq x \leq \pi.
\end{cases}
\]
(C)

The jump condition for this problem is given by
\[
\int_{\xi^-}^{\xi^+} G'' + k^2 G \, dx = \int_{\xi^-}^{\xi^+} \delta(x - \xi) \, dx
\]
\[
G'('\xi^+') - G'('\xi^-') = 1,
\]
where we have used the fact that \( G \) is continuous at \( x = \xi \). Then substituting in (C), we have
\[
C k \cos k\pi \xi \sin k(\pi - \xi) + C k \sin k\pi \xi \cos k(\pi - \xi) = 1
\]
\[
C = \frac{1}{k \sin k\pi},
\]
where we have used a trigonometric identity. Hence we have that
\[
G = \frac{\cos k\pi x_< \cos k(\pi - x_>)}{k \sin k\pi},
\]
which breaks down when \( k = n \).