1. (8 points) Consider the following equation on the entire \( x-y \) plane:

\[
\frac{\partial^2 \rho}{\partial x^2} - y^2 \frac{\partial^2 \rho}{\partial y^2} + \frac{\partial \rho}{\partial x} + y = 0.
\]

Classify it and transform it into canonical form. Discuss any singularities that occur in the transformed equations. The domain for each is the entire \( x-y \) plane.

**Solution.** Using the standard notation for the coefficients, we have \( A = 1, B = 0, \) and \( C = -y^2. \) Therefore, we obtain

\[
\frac{dy}{dx} = \frac{B \pm \sqrt{B^2 - 4AC}}{2A} = \pm \frac{\sqrt{4y^2}}{2} = \pm y
\]

\[
y = \xi e^x, \quad y = \eta e^{-x} \quad \implies \quad y^2 = \xi \eta, \quad y = \text{sgn}(\xi) \sqrt{\xi \eta}
\]

\[
\xi = ye^{-x}, \quad \eta = ye^x.
\]

Thus we have that the equation is hyperbolic except along the \( x \)-axis, in which case it is already in parabolic canonical form. Transforming into canonical form for the case \( y \neq 0, \) we obtain

\[
\partial_x = \frac{\partial \xi}{\partial x} \frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial x} \frac{\partial}{\partial \eta} = -ye^{-x} \frac{\partial}{\partial \xi} + ye^x \frac{\partial}{\partial \eta} = -\xi \frac{\partial}{\partial \xi} + \eta \frac{\partial}{\partial \eta}
\]

\[
\partial_y = \frac{\partial \xi}{\partial y} \frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial y} \frac{\partial}{\partial \eta} = e^{-x} \frac{\partial}{\partial \xi} + e^x \frac{\partial}{\partial \eta} = \frac{1}{y} \left( \xi \frac{\partial}{\partial \xi} + \eta \frac{\partial}{\partial \eta} \right)
\]

\[
\begin{align*}
\left( -\xi \frac{\partial}{\partial \xi} + \eta \frac{\partial}{\partial \eta} \right)^2 \rho - y^2 \frac{\partial}{\partial y} \left[ \frac{1}{y} \left( \xi \frac{\partial \rho}{\partial \xi} + \eta \frac{\partial \rho}{\partial \eta} \right) \right] + \left( -\xi \frac{\partial \rho}{\partial \xi} + \eta \frac{\partial \rho}{\partial \eta} \right) + y = 0 \\
\left( \xi \frac{\partial}{\partial \xi} - \eta \frac{\partial}{\partial \eta} \right)^2 \rho + \left( \xi \frac{\partial \rho}{\partial \xi} + \eta \frac{\partial \rho}{\partial \eta} \right) - \frac{y}{y} \left( \xi \frac{\partial \rho}{\partial \xi} + \eta \frac{\partial \rho}{\partial \eta} \right)^2 \rho + \left( -\xi \frac{\partial \rho}{\partial \xi} + \eta \frac{\partial \rho}{\partial \eta} \right) + y = 0 \\
-4\eta \xi \frac{\partial^2 \rho}{\partial \xi \partial \eta} + 2\eta \frac{\partial \rho}{\partial \eta} + \text{sgn}(\xi) \sqrt{\xi \eta} = 0 \\
\frac{\partial^2 \rho}{\partial \xi \partial \eta} - \frac{1}{2} \frac{\partial \rho}{\partial \eta} - \frac{\text{sgn}(\xi)}{4 \sqrt{\xi \eta}} = 0.
\end{align*}
\]

The only singularities are along the \( \xi \)- and \( \eta \)-axes. But these are never reached, since by (A) they correspond to infinite values of \( x \) when \( y \neq 0. \)
2. (8 points) Consider the canonical form of the elliptic equation:

$$
\frac{\partial^2 \phi}{\partial \xi^2} + \frac{\partial^2 \phi}{\partial \eta^2} + \alpha \frac{\partial \phi}{\partial \xi} + \beta \frac{\partial \phi}{\partial \eta} + \gamma \phi + \delta = 0,
$$

where all the coefficients are constant. Introduce appropriate substitution(s) to reduce (1.1a) to

$$
\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} + \epsilon u = 0,
$$

and identify the value of $\epsilon$.

Solution. Letting $\phi(\xi, \eta) = w(\xi, \eta) - \delta/\gamma$, we obtain

$$
\frac{\partial^2 w}{\partial \xi^2} + \frac{\partial^2 w}{\partial \eta^2} + \alpha \frac{\partial w}{\partial \xi} + \beta \frac{\partial w}{\partial \eta} + \gamma \left( w - \frac{\delta}{\gamma} \right) + \delta = 0
$$

$$
\frac{\partial^2 w}{\partial \xi^2} + \frac{\partial^2 w}{\partial \eta^2} + \alpha \frac{\partial w}{\partial \xi} + \beta \frac{\partial w}{\partial \eta} + \gamma w = 0,
$$

which takes care of the constant forcing. Then we let $w(\xi, \eta) = e^{a\xi+b\eta} u(\xi, \eta)$, for some constants $a$ and $b$. Substituting this expression into the above, we have

$$
\left( a^2 e^{a\xi+b\eta} u + 2ae^{a\xi+b\eta} \frac{\partial u}{\partial \xi} + e^{a\xi+b\eta} \frac{\partial^2 u}{\partial \xi^2} \right) + \left( b^2 e^{a\xi+b\eta} u + 2be^{a\xi+b\eta} \frac{\partial u}{\partial \eta} + e^{a\xi+b\eta} \frac{\partial^2 u}{\partial \eta^2} \right)
$$

$$
+ \alpha \left( ae^{a\xi+b\eta} u + e^{a\xi+b\eta} \frac{\partial u}{\partial \xi} \right) + \beta \left( be^{a\xi+b\eta} u + e^{a\xi+b\eta} \frac{\partial u}{\partial \eta} \right) + \gamma e^{a\xi+b\eta} u = 0.
$$

Cancelling the exponentials, we have

$$
\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} + (2a + \alpha) \frac{\partial u}{\partial \xi} + (2b + \beta) \frac{\partial u}{\partial \eta} + (a^2 + b^2 + \alpha a + \beta b + \gamma) u = 0.
$$

Thus, by setting $a = -\alpha/2$, $b = -\beta/2$, we obtain

$$
\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} + \left( \frac{\alpha^2 + \beta^2}{4} - \frac{\alpha^2 + \beta^2}{2} + \gamma \right) u = 0
$$

$$
\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} + \left( \gamma - \frac{\alpha^2 + \beta^2}{4} \right) u = 0,
$$

which is in the form of (1.1b) with $\epsilon = \gamma - (\alpha^2 + \beta^2)/4$.

3. (8 points) Consider the following problem in polar coordinates $(r, \theta)$:

$$
\frac{\partial^2 f}{\partial r^2} + \frac{1}{r} \frac{\partial f}{\partial r} + \frac{1}{r^2} \frac{\partial^2 f}{\partial \theta^2} = 0.
$$

If the domain of the problem is an annulus, use separation of variables to find the particular solutions (eigenfunctions) of the problem.
Solution. We let \( f(r, \theta) = R(r)\Theta(\theta) \) in the above to obtain

\[
\begin{align*}
R'' &+ \frac{R'}{r} \Theta + \frac{R}{r^2} \Theta'' = 0 \\
\frac{R''}{R} + \frac{R'}{Rr} + \frac{1}{r^2} \frac{\Theta''}{\Theta} &= 0 \\
r^2 \left( \frac{R''}{R} + \frac{R'}{Rr} \right) &= -\frac{\Theta''}{\Theta}.
\end{align*}
\]

The left-hand side is a function of \( r \) only; the right-hand side is a function of \( \theta \) only. Therefore, both must be constant, which we shall denote by \( \lambda^2 \). We focus on the \( \theta \) equation and note that since the domain is in annulus, the solution must be continuous at \( \theta = 0 \) and \( \theta = 2\pi \). Thus we have

\[
-\frac{\Theta''}{\Theta} = \lambda^2, \quad \Theta(0) = \Theta(2\pi) \\
\Theta'' + \lambda^2 \Theta = 0 \\
\Theta(\theta) = a\sin \lambda \theta + a_c \cos \lambda \theta.
\]

Due to the periodicity conditions, we must have \( \lambda = \pm n \). Hence the \( R \) equation becomes the Euler equation

\[
R'' + \frac{R'}{r} = \frac{n^2}{r^2} R \\
R = r^\alpha \\
\Rightarrow \quad \alpha(\alpha - 1) + \alpha = n^2 \\
(\alpha - n)(\alpha + n) = 0 \\
R(r) = a_+ r^n + a_- r^{-n}
\]

\[
\phi_n(r, \theta) = a_1 r^n \sin n\theta + a_2 r^n \cos n\theta + a_3 r^{-n} \sin n\theta + a_4 r^{-n} \cos n\theta, \quad n \neq 0.
\]

For the case \( n = 0 \), we obtain the double root \( \alpha^2 = 0 \), which implies the addition of a logarithm term. In addition, we see that the sine term vanishes and the cosine term becomes a constant, so we have

\[
\phi_0(r, \theta) = a_1 + a_2 \log r.
\]

Since we are working in an annulus, \( r \) is always bounded away from zero, so all the terms listed are allowable.

4. We wish to define a set of polynomials \( \{T_n(x)\}_{n=0}^\infty \) for \( x \in [-1,1] \) that have the following properties:

(i) \( T_n(x) = \sum_{i=0}^n a_{in} x^i \) is a polynomial of degree \( n \) in \( x \),

(ii) the set \( \{T_n(x)\}_{n=0}^\infty \) is orthogonal with weight function \( \rho(x) = (1-x^2)^{-1/2} \), and
(iii) \( T_n(1) = 1. \)

We wish to prove such a definition is unique using induction.

(a) (2 points) Anchor the induction by constructing \( T_0(x) \) and \( T_1(x) \).

Solution. \( T_0(x) \) is the constant polynomial whose value at \( x = 1 \) is 1, so \( T_0(x) = 1. \) Since \( ||T_0|| \neq 0 \), we see that

\[
\langle T_1, T_0 \rangle = \langle a_{11}x + a_{01}, T_0 \rangle = a_{11}\langle x, 1 \rangle + a_{01}||T_0||^2 = a_{01}||T_0||^2 = 0,
\]

where we have used the fact that \( x \rho \) is odd to eliminate the first inner product. But this implies that \( a_{01} = 0 \), which implies that \( a_{11} = 1 \) to satisfy condition (iii). Therefore, \( T_1(x) = x \).

(b) (4 points) Show that if \( T_n(x) \) and \( T^*_n(x) \) both satisfy the definitions above, then \( T_n(x) = T^*_n(x) \).

Solution. In order to calculate the coefficients \( a_{in} \), we must satisfy the following \( n \) orthogonality conditions:

\[
\langle T_n, T_m \rangle = \sum_{i=0}^{n} a_{in} \langle x^i, T_m \rangle = 0, \quad m = 0, 1, \ldots, n - 1, \quad (A.1)
\]

as well as the condition that

\[
T_n(1) = \sum_{i=0}^{n} a_{in} = 1. \quad (A.2)
\]

This is a set of \( n + 1 \) linear equations in the \( n + 1 \) unknowns \( \{a_{in}\}_{0}^{n} \). Therefore, there is either exactly one solution or an infinite number of solutions. If there as infinite number, then there exists another solution

\[
y(x) = \sum_{i=0}^{n} y_n x^i
\]

which satisfies (A.1) and \( y(1) = 0. \) (This is an element of the null space.) Here \( y_n \neq 0 \) because if it did, we would have infinitely many solutions of degree \( n - 1 \), which violates the induction assumption. Let \( T^*_n(x) = T_n(x) - a_{nn}y(x)/y_n \). Since \( T^*_n \) is a linear combination of \( T_n \) and \( y_n \), it too must satisfy the orthogonality and \( T^*_n(1) = 1 \) conditions. But \( T^*_n(x) \) is also of degree \( n - 1 \), which is impossible. Therefore, \( T_n(x) \) is unique.

(c) (3 points) Verify that \( T_n(x) = \cos(n \cos^{-1} x) \).

Solution. By basic complex variables, we have that

\[
T_n(x) = \cos(n \cos^{-1} x) = \Re \left( e^{in \cos^{-1} x} \right) = \Re \left( e^{i \cos^{-1} x} \right)^n = \Re \left( x \pm i \sqrt{1 - x^2} \right)^n.
\]

The real part of the right-hand side must be made up of the sum of products of the form

\[
x^{n-2j}(1 - x^2)^j, \quad j = 0, 1, \ldots, \left\lfloor \frac{n}{2} \right\rfloor,
\]
which is a polynomial in \( x \). It is a polynomial of degree \( n \) due to the \( j = 0 \) term. Clearly \( T_n(1) = \cos(n0) = 1 \). To show orthogonality, we let \( x = \cos \theta \) in the integral:

\[
\langle T_m, T_n \rangle = \int_{-1}^{1} \frac{\cos(m \cos^{-1} x) \cos(n \cos^{-1} x)}{\sqrt{1 - x^2}} \, dx = \int_{\pi}^{0} \frac{\cos(n \theta) \cos(m \theta)}{\sin \theta} (-\sin \theta \, d\theta)
\]

\[
= \frac{1}{2} \int_{-\pi}^{\pi} \cos(n \theta) \cos(m \theta) \, d\theta = 0
\]

by the orthogonality of Fourier series (as shown in class).

5. (5 points) Show that the set of functions \( \phi_{n1} = \sin(n \pi x / L) \) is orthogonal respect to the inner product

\[
\langle f, g \rangle = \int_{-L}^{L} f g \, dx
\]

and each has length \( \sqrt{L} \).

Solution.

\[
\langle \phi_{m1}, \phi_{n1} \rangle = \int_{-L}^{L} \sin \left( \frac{m \pi x}{L} \right) \sin \left( \frac{n \pi x}{L} \right) \, dx
\]

\[
= \frac{1}{2} \int_{-L}^{L} \cos \left( \frac{(m - n)\pi x}{L} \right) - \cos \left( \frac{(m + n)\pi x}{L} \right) \, dx
\]

\[
= \frac{1}{2} \left[ \frac{L}{(m - n)\pi} \sin \left( \frac{(m - n)\pi x}{L} \right) - \frac{L}{(m + n)\pi} \sin \left( \frac{(m + n)\pi x}{L} \right) \right]_{-L}^{L} = 0,
\]

whenever \( m \neq n \). In the case that \( m = n \), we have

\[
\langle \phi_{n1}, \phi_{n1} \rangle = \frac{1}{2} \int_{-L}^{L} 1 - \cos \left( \frac{2n \pi x}{L} \right) \, dx
\]

\[
||\phi_{n1}||^2 = \frac{1}{2} (2L) = L.
\]