ODE Review

First-Order Equations

First-order equations come in two flavors that we can easily solve. First, a *separable equation* is one where the dependent and independent variables may be separated so that the equation may be integrated:

$$\begin{aligned} \dot{y} &= ty^3 \\ \frac{dy}{y^3} &= t\,dt \\ -\frac{1}{2y^2} &= \frac{t^2}{2} + C. \end{aligned}$$

Next, for a linear equation

$$\dot{y} + p(t)y = g(t),$$

one may multiply by an *integrating factor* $\exp(\int p\,dt)$ to obtain an integrable right-hand side, as in this example:

$$\begin{aligned} \dot{y} - 2ty &= \frac{2}{\sqrt{\pi}}e^{t^2} \\ e^{-t^2}\dot{y} - 2te^{-t^2}y &= \frac{2}{\sqrt{\pi}} \\ \frac{d}{dt}\left(e^{-t^2}y\right) &= \frac{2}{\sqrt{\pi}} \\ y &= e^{t^2}\left(\frac{2t}{\sqrt{\pi}} + C\right). \end{aligned}$$

Second-Order Equations

Similarly, second-order equations come in two flavors that we can easily solve. For a linear equation with constant coefficients, we may substitute $e^{\lambda t}$ to obtain a quadratic equation for λ:

$$a\ddot{y} + b\dot{y} + cy = 0 \qquad \Longrightarrow \qquad a\lambda^2 + b\lambda + c = 0.$$

Each λ gives rise to a solution except in the case of repeated roots.

Next, for an *Euler equation*

$$at^2\ddot{y} + bt\dot{y} + cy = 0,$$

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substituting t^λ again yields a quadratic equation for λ:

$$a\lambda(\lambda - 1) + b\lambda + c = 0.$$

Again, each λ gives rise to a solution except in the case of repeated roots.

If the equation has a nonzero right-hand side, there are two methods to find the particular solution. If we have constant coefficients, one may use the *method of undetermined coefficients*. In this method, one tries an unknown multiple of the right-hand side and then tries to find the coefficient that satisfies the equation. For instance:

$$y_\mathrm{p} = Ae^{2t} \qquad \Longrightarrow \qquad \begin{aligned} \ddot{y} - y &= e^{2t} \\ 4A - A &= 1 \\ y_\mathrm{p} &= \frac{e^{2t}}{3}. \end{aligned}$$

Alternatively, one may use the *variation of parameters* formula:

$$y_\mathrm{p} = -y_1 \int \frac{y_2 g}{W}\, dt + y_2 \int \frac{y_1 g}{W}\, dt,$$

where $\{y_1, y_2\}$ is a fundamental set of solutions to the homogeneous equation, g is the right-hand side, and W is the *Wronskian* $y_1\dot{y}_2 - \dot{y}_1 y_2$. In the case above, substituting $e^{\lambda t}$ into the homogeneous equation yields $\lambda = \pm 1$, so let $y_1 = e^t$, $y_2 = e^{-t}$. Then $W = -2$ and the formula becomes

$$\begin{aligned} y_\mathrm{p} &= -e^t \int \frac{e^{-t}e^{2t}}{(-2)}\, dt + e^{-t} \int \frac{e^t e^{2t}}{(-2)}\, dt, \\ &= e^t \left(\frac{e^t}{2}\right) - e^{-t}\left(\frac{e^{3t}}{6}\right) = \frac{e^{2t}}{3}, \end{aligned}$$

so the answers match.

For more details, a good place to start is the book *Elementary Differential Equations and Boundary Value Problems* by Boyce and DiPrima, which we use on campus in our undergraduate courses.