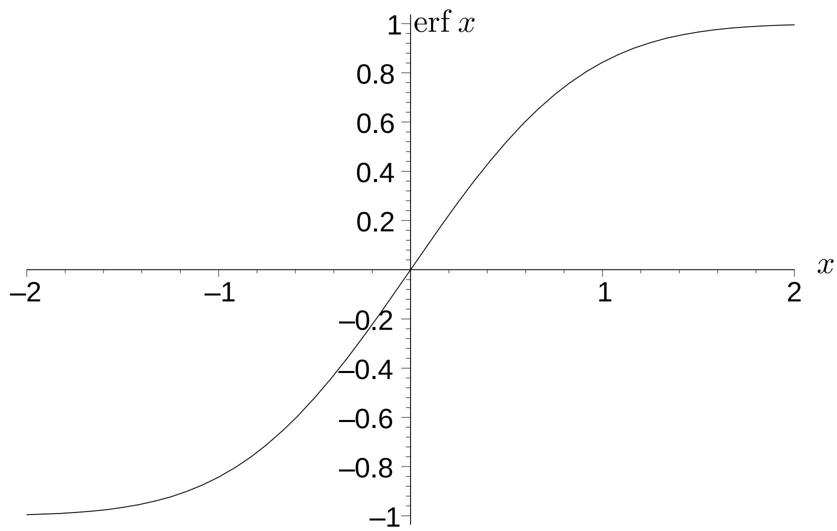
The Error Function

The *error function* is defined by

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-z^2}\, dz.$$

It arises often in probability problems. Also, because of the relationship between the random walk and diffusion processes, it occurs often in diffusion problems.

The coefficient in front of the integral normalizes $\operatorname{erf}(\infty) = 1$. Clearly $\operatorname{erf} x$ is odd in x. The error function is plotted below.



$\operatorname{erf} x$ *vs.* x.

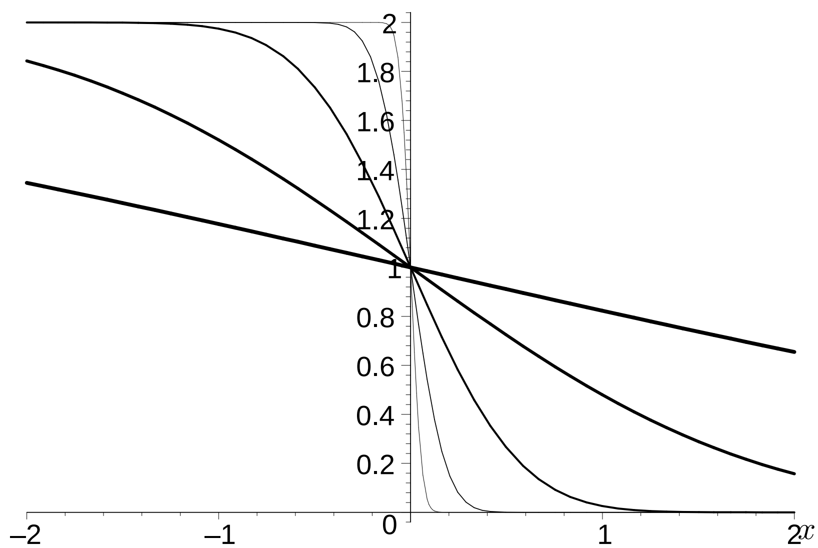
The *complementary error function* is defined by

$$\begin{aligned}\operatorname{erfc}(x) = 1 - \operatorname{erf}(x) &= \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-z^2}\,dz - \frac{2}{\sqrt{\pi}} \int_0^x e^{-z^2}\,dz \\ &= \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-z^2}\,dz.\end{aligned}$$

In the example in class, we found that the solution to a diffusion equation was given by

$$\operatorname{erfc}\left(\frac{x}{2\sqrt{t}}\right).$$

(We also found this same solution by similarity techniques earlier in the course.) This function is plotted below for various values of t. Note that increasing t doesn't change the *shape* of the curve; it just changes the *scale*.



$\operatorname{erfc}(x/2\sqrt{t})$ *vs.* x. In increasing order of thickness: $t = 0.001, 0.01, 0.1, 1, 10$.