Envelopes

For the problem
\[
\frac{\partial \rho}{\partial t} + \rho \frac{\partial \rho}{\partial x} = 0, \quad \rho(x, 0) = -\tanh x,
\]
we determined that the solution is given by
\[
\rho = -\tanh \xi \quad (1a)
\]
along characteristics given by
\[
x = \xi - (\tanh \xi)t. \quad (1b)
\]
We also determined that \( t_B = 1 \).

\[\rho(x, t) \text{ vs. } x \] as given by (1) with \( t = 0, 0.5, 1, 1.5, 2 \) (in increasing order of thickness). Note the single vertical tangent at \( t_B = 1 \). The dashed and dotted lines show the positions of the vertical tangents at \( t = 1.5 \) and \( t = 2 \). Note these will correspond to the envelope.
We also determined that the envelope of intersecting characteristics is given by (1b) and

\[ 0 = 1 - (\text{sech}^2 \xi) t. \]  

(2)

Below are plotted characteristics given by (2) for \( \xi = -0.8 \) through \( \xi = 0.8 \) incremented by 0.1, as well as the envelope (thick line). Note that the envelope has a cusp at \( t_B \).

Characteristics given by (1b) (thin lines) and envelope given by (1b) and (2) (thick line). The dashed and dotted lines illustrate the position of the envelope at \( t = 1.5 \) and \( t = 2 \), for comparison with the diagram on reverse.