Supplemental Study Material

Solutions (Revised)

1. Let \( f(x; \epsilon) = o(\phi(x; \epsilon)) \) uniformly for \( x \in [0, 1] \). Here \( f \) and \( \phi \) are continuous for \( x \in [0, 1] \) and (at the very least) \( \epsilon > 0 \).

(a) Show that
\[
\int_0^\epsilon f(x; t) \, dt = o\left( \int_0^\epsilon |\phi(x; t)| \, dt \right)
\]
uniformly in \([0, 1]\).

Solution. By the definition of the limit, \( f(x; \epsilon) = o(\phi(x; \epsilon)) \) is equivalent to stating that for any \( \delta > 0 \), there exists an \( \epsilon^* \) such that
\[
\left| \frac{f(x; \epsilon)}{\phi(x; \epsilon)} \right| \leq \delta \quad \implies \quad |f(x; \epsilon)| \leq \delta |\phi(x; \epsilon)|, \quad 0 < \epsilon < \epsilon^*.
\]
The uniformity implies that \( \delta \) is independent of \( x \). Thus we have
\[
\int_0^\epsilon f(x; t) \, dt \leq \int_0^\epsilon |f(x; t)| \, dt \leq \int_0^\epsilon \delta |\phi(x; t)| \, dt = \delta \int_0^\epsilon |\phi(x; t)| \, dt = \delta \left| \int_0^\epsilon |\phi(x; t)| \, dt \right|
\]
where now \( \epsilon \) corresponds to \( \epsilon^* \). In addition, we have
\[
\int_0^\epsilon |f(x; t)| \, dt \geq \left| \int_0^\epsilon f(x; t) \, dt \right| \quad \implies \quad \left| \int_0^\epsilon f(x; t) \, dt \right| \leq \delta \left| \int_0^\epsilon |\phi(x; t)| \, dt \right|
\]
which is simply the relationship in (A) defining the \( o \) relationship between the integrals.

(b) Show that (S.1) does not hold if we remove the absolute value from the right-hand side.

Solution. Here’s a typical counterexample. Let
\[
\phi(x; \epsilon) = \sin(\epsilon - x), \quad f(x; \epsilon) = \epsilon \phi^2.
\]
Clearly \( f \) and \( \phi \) satisfy the continuity assumptions. Then
\[
\lim_{\epsilon \to 0^+} \frac{f(x; \epsilon)}{\phi(x; \epsilon)} = \lim_{\epsilon \to 0^+} \epsilon \sin(\epsilon - x) = 0
\]
uniformly in [0, 1]. Computing the integrals in (S.1) (without the absolute value), we have
\[
\int_0^\epsilon f(x; t) \, dt = \int_0^\epsilon t \sin^2(t - x) \, dt > 0, \quad \epsilon > 0, \\
\int_0^\epsilon \phi(x; t) \, dt = [-\cos(\epsilon - x)]_0^\epsilon = \cos x - \cos(\epsilon - x).
\]
Therefore, we see that if we choose \( x = \epsilon/2 \), the integral on the the right is zero, and hence the relationship does not hold uniformly in \( x \).

2. Consider the following equation:
\[
\tan z = z. \tag{S.2}
\]
(a) Show that for all integral \( n \), there exists exactly one root \( z_n \) of (S.2) in the region \(( (n - 1/2)\pi, (n + 1/2)\pi) \).

**Solution.** In each region \(( (n - 1/2)\pi, (n + 1/2)\pi) \), \( \tan z \) takes on all possible values, so there must be at least one root. To check the uniqueness, we note that
\[
\frac{d(\tan z)}{dz} = \sec^2 z \geq 1 = \frac{d(z)}{dz}.
\]
Therefore, we see that the derivative of the left-hand side is always larger than the derivative of the right-hand side. Thus, the left-hand side will always be larger than the right in each region for \( z \) greater than the point of intersection, and hence there will be exactly one such point.

(b) Find an asymptotic expansion for \( z_n \) as \( n \to \infty \). Include terms up to \( O(n^{-2}) \).

**Solution.** From the right-hand side, we see that \( z_n \to \infty \), so we are looking for large values of \( \tan z \), which occurs as \( z \to (n + 1/2)\pi^- \). Therefore, we see that as \( n \to \infty \), \( z_n \sim (n + 1/2)\pi - z_* \), where \( z_* \to 0^+ \) as \( n \to \infty \). Substituting in this form, we have
\[
\sin((n + 1/2)\pi - z_*) = (n + 1/2)\pi - z_* \\
(-1)^n \cos z_* = [(n + 1/2)\pi - z_*][( -1)^n \sin z_*] \\
\cos z_* = [(n + 1/2)\pi - z_*] \sin z_*.
\]
Now we let
\[
z_* = \frac{z_1}{n} + \frac{z_2}{n^2} + o(n^{-2}), \quad n \to \infty,
\]
to obtain, to leading orders,
\[
\cos \left( \frac{z_1}{n} + \frac{z_2}{n^2} \right) = \left[ \left( n + \frac{1}{2} \right) \pi - \left( \frac{z_1}{n} + \frac{z_2}{n^2} \right) \right] \sin \left( \frac{z_1}{n} + \frac{z_2}{n^2} \right) \\
1 - \frac{1}{2} \left( \frac{z_1}{n} \right)^2 = \left[ \left( n + \frac{1}{2} \right) \pi \right] \left( \frac{z_1}{n} + \frac{z_2}{n^2} \right) \\
\frac{1}{\pi} = z_1 + \frac{z_2 + z_1/2}{n} \\
z_1 = \frac{1}{\pi}, \quad z_2 = -\frac{1}{2\pi} \\
z_n \sim n\pi + \frac{\pi}{2} - \frac{1}{n\pi} + \frac{1}{2n^2\pi}.
\]