Homework Set 4 Solutions

1. Consider the following model for an epidemic:

\[
\frac{d\tilde{S}}{dt} = b_1 N - \beta \tilde{S} \tilde{I} - b_2 \tilde{S}, \tag{4.1a}
\]
\[
\frac{d\tilde{I}}{dt} = \beta \tilde{S} \tilde{I} - (b_2 + r) \tilde{I}, \tag{4.1b}
\]
\[
\frac{d\tilde{M}}{dt} = r \tilde{I} - b_2 \tilde{M}, \tag{4.1c}
\]

where \( \tilde{S}, \tilde{I}, \) and \( \tilde{M} \) are the populations of susceptible people, infected people, and immune people. \( N \) is the total population.

(a) (3 points) Interpret each of the constants \( \beta, b_1, b_2, \) and \( r \) biologically.

Solution. From (4.1a), we see that \( b_1 \) is the birth rate. Note that all three populations can reproduce, and all individuals are born susceptible. \( b_2 \) is the death rate, which is the same for each group. \( \beta \) is the infection rate, as it increases infectives when an infective and susceptible meet. \( r \) is the cure rate, as it increases immunes and reduces infectives.

Now consider the case where \( b_1 = b_2 = b. \)

(b) (3 points) Verify that \( N \) is a constant. Explain this phenomenon biologically.

Solution. Since \( N \) is the total population, we have that

\[
\frac{dN}{dt} = \frac{d(\tilde{S} + \tilde{I} + \tilde{M})}{dt} = bN - \beta \tilde{S} \tilde{I} - b\tilde{S} + \beta \tilde{S} \tilde{I} - (b + r) \tilde{I} + r \tilde{I} - b\tilde{M} \]
\[
= bN - b\tilde{S} - b\tilde{I} - b\tilde{M} = b(N - \tilde{S} - \tilde{I} - \tilde{M}) = 0.
\]

Therefore, the total population \( N \) must be constant. This occurs since the birth rate and death rate are the same.

(c) (4 points) Scale the equations using \( b^{-1} \) for the time scale.

Solution. We let

\[
S(t) = \frac{\tilde{S}(\tilde{t})}{N}, \quad I(t) = \frac{\tilde{I}(\tilde{t})}{N}, \quad M(t) = \frac{\tilde{M}(\tilde{t})}{N}, \quad \tilde{t} = b\tilde{t}
\]

in (4.1) to obtain

\[
\frac{dS}{dt} = 1 - \frac{\beta N}{b} SI - S, \tag{A.1}
\]
\[
\frac{dI}{dt} = \frac{\beta N}{b} SI - \left(1 + \frac{r}{b}\right) I = I \left[\frac{\beta N}{b} S - \left(1 + \frac{r}{b}\right)\right],
\]
\[
\frac{dM}{dt} = \frac{r}{b} I - M. \tag{A.2}
\]
(d) (4 points) Show that if $N < (b + r)/\beta$, the parasite cannot survive in the population.

Solution. We examine equation (A.1). We know that $0 \leq S \leq 1$, so the largest that the bracketed quantity can be is

$$\frac{\beta N}{b} - \left(1 + \frac{r}{b}\right).$$

If this quantity is negative, we have that $dI/dt$ is always negative and hence $I$ decays to zero, which then implies that $M = 0$ [from (A.2)] and $S = 1$. But this occurs whenever

$$\frac{\beta N}{b} < 1 + \frac{r}{b},$$

and after some manipulation, we obtain the desired result.

2. Consider the discrete map

$$x_{n+1} = F(x_n) = x_n + \lambda \cos 2\pi x_n, \quad 0 < \lambda < \frac{1}{2}, \quad -\frac{1}{2} \leq x \leq 1.$$ 

(a) (3 points) Find the fixed points of the map and their stability for various values of $\lambda$. Identify a critical value $\lambda_c$ where all solutions become unstable.

Solution. The fixed points $x_*$ have

$$x_* = x_* + \lambda \cos 2\pi x_*$$

$$2\pi x_* = \frac{(2m + 1)\pi}{2}$$

$$x_* = \frac{1}{4}, \frac{1}{4}, \frac{3}{4}, \frac{3}{4}$$

in the region under consideration. To check stability, we have that

$$F'(x_*) = 1 - 2\pi \lambda \sin 2\pi x_*,$$

$$F'(-1/4) = F'(3/4) = 1 + 2\pi \lambda > 1,$$

$$F'(1/4) = 1 - 2\pi \lambda.$$ 

Hence we see that $x_* = -1/4$ and $x_* = 3/4$ are always unstable, while $x_* = 1/4$ is stable for $\lambda < 1/\pi = \lambda_c$.

(b) (7 points) Show that a stable two-cycle exists for $\lambda \to \lambda_c^+$. (Hint: Let

$$\lambda = \lambda_c + \frac{\delta^2}{2\pi}, \quad x = x_c + \frac{\delta \bar{x}}{2\pi}, \quad 0 < \delta \ll 1,$$

where $\bar{x}$ is unknown and must be solved for at $O(\delta^3)$.)
Solution. Motivated by notes in class, we look for a two-cycle in the region $\lambda > 1/\pi$ where all three steady states are unstable. Hence we compute

$$F^2(x_n) = x_{n+1} + \lambda \cos 2\pi x_{n+1} = x_n + \lambda \cos 2\pi x_n + \lambda \cos 2\pi (x_n + \lambda \cos 2\pi x_n).$$

Calculating the fixed points, we have

$$x_* = x_* + \lambda \cos 2\pi x_* + \lambda \cos 2\pi (x_* + \lambda \cos 2\pi x_*)$$

$$\cos 2\pi (x_* + \lambda \cos 2\pi x_*) = -\cos 2\pi x_*.$$  \hspace{1cm} (C)

Note that the fixed points of $F$ also satisfy (C), as they should. In general, (C) cannot be solved analytically.

To find roots of this map in the neighborhood of the critical point, we recall that the two-cycle should emerge from the $x_*$ value that changes stability, so following the hint we let

$$x_* = \frac{1}{4} + \frac{\delta \bar{x}}{2\pi}, \quad \lambda = \frac{1}{\pi} + \frac{\delta^2}{2\pi}, \quad 0 < \delta \ll 1.$$

Note that the form of the $\lambda$ expansion guarantees that we are examining the values of $\lambda$ where all three fixed points of $F$ are unstable.

Substituting the above into (C) and expanding to $O(\delta^3)$, we have

$$\cos \left( \frac{\pi}{2} + \delta \bar{x} + 2\pi \left( \frac{1}{\pi} + \frac{\delta^2}{2\pi} \right) \cos \left( \frac{\pi}{2} + \delta \bar{x} \right) \right) = -\cos \left( \frac{\pi}{2} + \delta \bar{x} \right)$$

$$- \sin \left( \delta \bar{x} - (2 + \delta^2) \sin \delta \bar{x} \right) = \sin \delta \bar{x}$$

$$\sin \left( -\delta \bar{x} + (2 + \delta^2) \left( \delta \bar{x} - \frac{\delta^3 \bar{x}^3}{6} \right) \right) = \delta \bar{x} - \frac{\delta^3 \bar{x}^3}{6}$$

$$\sin \left( \delta \bar{x} + 2\delta^3 \bar{x} - \frac{\delta^3 \bar{x}^3}{3} \right) = \delta \bar{x} - \frac{\delta^3 \bar{x}^3}{6}$$

$$\delta \bar{x} + \delta^3 \bar{x} - \frac{\delta^3 \bar{x}^3}{3} - \frac{\delta^3 \bar{x}^3}{6} = \delta \bar{x} - \frac{\delta^3 \bar{x}^3}{6}$$

$$3\bar{x} = \bar{x}^3.$$

The solution $\bar{x} = 0$ corresponds to $x_* = 1/4$. The other two solutions $\bar{x}_\pm = \pm \sqrt{3}$ correspond to two solutions which form a two cycle. Substituting this form into (B), we have

$$F'(x_+)F'(x_-) = \left[ 1 - 2\pi \left( \frac{1}{\pi} + \frac{\delta^2}{2\pi} \right) \sin 2\pi \left( \frac{1}{4} + \frac{\delta \sqrt{3}}{2\pi} \right) \right] \times$$

$$\left[ 1 - 2\pi \left( \frac{1}{\pi} + \frac{\delta^2}{2\pi} \right) \sin 2\pi \left( \frac{1}{4} - \frac{\delta \sqrt{3}}{2\pi} \right) \right]$$

$$= \left[ 1 - (2 + \delta^2) \sin \left( \frac{\pi}{2} + \delta \sqrt{3} \right) \right] \left[ 1 - (2 + \delta^2) \sin \left( \frac{\pi}{2} - \delta \sqrt{3} \right) \right]$$

$$= \left[ 1 - (2 + \delta^2) \cos(\delta \sqrt{3}) \right] \left[ 1 - (2 + \delta^2) \cos(\delta \sqrt{3}) \right]$$

$$= \left[ 1 - (2 + \delta^2) \left( 1 - \frac{3\delta^2}{2} \right) \right]^2 = (-1 + 2\delta^2)^2.$$
As this quantity has absolute value less than 1, the two-cycle is stable.

3. A species of insect (population $\tilde{N}_t$) is controlled by introducing a constant sterile insect population $\tilde{S}$. The resulting evolution equation is given by

$$\tilde{N}_{t+1} = \frac{(R + 1)\tilde{N}_t^2}{RN_t^2/M + \tilde{N}_t + \tilde{S}}, \quad R > 0, \quad M > 0. \quad (4.3)$$

(a) (2 points) Introduce a proper scaling for the populations, and normalize (4.3).

Solution. Upon consideration of (4.3), we see that $R$ must be dimensionless and $M$ must have dimensions of population. Therefore, we let

$$\tilde{N}_t = MN_t, \quad \tilde{S} = MS.$$ 

Making these substitutions into (4.3), we obtain

$$MN_{t+1} = \frac{(R + 1)M^2N^2_t}{RM^2N_t^2/M + MN_t + MS},$$

$$N_{t+1} = F(N_t), \quad F(N_t) = \frac{(R + 1)N_t^2}{RN_t^2 + N_t + S}. \quad (D)$$

(b) (3 points) Show that there are three steady states only if $S < R/4$.

Solution. From (D), we have that the steady states $N$ are given by

$$N = \frac{(R + 1)N^2}{RN^2 + N + S}$$

$$N(RN^2 + N + S) = (R + 1)N^2$$

$$N(RN^2 - RN + S) = 0$$

$$N_s = 0, \quad RN^2 - RN + S = 0$$

$$N_{\pm} = \frac{R \pm \sqrt{R^2 - 4RS}}{2R} = \frac{1 \pm \sqrt{1 - 4S/R}}{2}. \quad (E.1)$$

We note that $N_{\pm}$ will be real only if $1 - 4S/R > 0$, or $S < R/4$. In that case, the square root is smaller than 1, so both of $N_{\pm}$ are positive.

(c) (4 points) Verify that if $S > R/4$, the population will be driven to zero.

Solution. Computing $F'(N_t)$ from (D), we have

$$F'(N_t) = \frac{2(R + 1)N_t}{RN_t^2 + N_t + S} - \frac{(R + 1)N_t^2(2RN_t + 1)}{(RN_t^2 + N_t + S)^2}$$

$$= \frac{(R + 1)N_t^2(2RN_t^2 + N_t + S) - (2RN_t + 1)N_t}{(RN_t^2 + N_t + S)^2} = \frac{(R + 1)N_t(N_t + 2S)}{(RN_t^2 + N_t + S)^2}$$

$$F'(0) = 0,$$
so the origin is always stable. Therefore, if there are no other steady states (i.e., when \( S > R/4 \)), the population must be driven to zero.

(d) (5 points) Find the stability of the other steady states, and sketch a diagram of the steady states vs. \( S \).

**Solution.** Computing \( F'(N_t) \) for the other steady states, we have

\[
F'(N_{\pm}) = \frac{(R + 1)N_{\pm}(N_{\pm} + 2S)}{(RN_{\pm}^2 + N_{\pm} + S)^2} = \frac{(R + 1)N_{\pm}(N_{\pm} + 2S)}{[(R + 1)N_{\pm}]^2} = \frac{(N_{\pm} + 2S)}{(R + 1)N_{\pm}},
\]

where we have used (E.1). This is always positive, so we must determine when it is less than 1:

\[
(N_{\pm} + 2S) < (R + 1)N_{\pm}
\]

\[2S < R \left( \frac{1 \pm \sqrt{1 - 4S/R}}{2} \right)\]

\[\frac{4S}{R} < 1 \pm \sqrt{1 - \frac{4S}{R}}\]

\[\pm \sqrt{1 - \frac{4S}{R}} > \frac{4S}{R} - 1.\]

When the steady state exists, the right-hand side is negative, so the expression is true for the positive square root, so \( N_+ \) is stable. Continuing for \( N_- \), we have (using the fact that the right-hand side is negative)

\[1 - \frac{4S}{R} < \left( \frac{4S}{R} - 1 \right)^2\]

\[\left( 1 - \frac{4S}{R} \right) \left[ 1 - \left( 1 - \frac{4S}{R} \right) \right] = \frac{4S}{R} \left( 1 - \frac{4S}{R} \right) < 0.\]

But this is false, since the parenthetical quantity is positive. So \( N_- \) is unstable. A graph is shown on the next page. Here the solid lines are stable, and the dotted lines are unstable.

(e) (2 points) Explain biologically the possible behaviors if \( 0 < S < R/4 \).

**Solution.** From the diagram, we expect that if the initial value of \( N \) is small, the population will decay away to zero, while if it is higher, then it will increase to the steady state \( N_+ \). This makes sense, since with a smaller number of sterile individuals set free to breed, the insect population could still increase.