Homework Set 1 Solutions

1. The Navier-Stokes equations are the governing equations for much of fluid mechanics. In dimensional form, they are given by

\[ \rho \left( \frac{\partial \tilde{u}}{\partial \tilde{t}} + \tilde{u} \cdot \nabla \tilde{u} \right) = -\nabla \tilde{p} + \mu \nabla^2 \tilde{u}, \]

where \( \rho \) is the density, \( \tilde{u} \) is the velocity, \( \tilde{p} \) is the pressure, and \( \mu \) is the bulk viscosity. The \( \nabla \) indicates that the spatial derivatives are taken with respect to variables with dimensions.

(a) (4 points) Write the units of each of the quantities listed in the Navier-Stokes equations. Check your work by verifying that each of the terms in the equation have the same units.

Solution.

\[ [\rho] = ML^{-3}, \quad [\tilde{u}] = LT^{-1}, \quad [\tilde{t}] = T, \quad [\tilde{p}] = ML^{-1}T^{-2}, \quad [\mu] = ML^{-1}T^{-1}. \]

To check our work, we substitute these expressions and find that

\[ \left[ \rho \frac{\partial \tilde{u}}{\partial \tilde{t}} \right] = \left[ \rho \tilde{u} \cdot \nabla \tilde{u} \right] = \left[ \nabla \tilde{p} \right] = \left[ \mu \nabla^2 \tilde{u} \right] = \frac{ML}{L^3T^2}. \]

(b) (5 points) Given a characteristic velocity \( U \) and length scale \( L \), scale the equations. On how many dimensionless parameters does your solution depend?

Solution. Divide the relevant length scales by \( L \) and let

\[ \tilde{u} = Uu, \quad \tilde{t} = \frac{L}{U}t, \quad \tilde{p} = \rho U^2 p = \frac{\mu U}{L}q. \]

Note there are two possible choices for the nondimensionalization of \( \tilde{p} \). However, since \( U \) was given as our characteristic velocity, we should normalize \( \tilde{u} \) by it, rather than by other combinations of our parameters. Using these choices, we have

\[ \frac{\rho U^2}{L} \left( \frac{\partial u}{\partial \tilde{t}} + u \cdot \nabla u \right) = -\frac{\rho U^2}{L} \nabla p + \frac{\mu U}{L^2} \nabla^2 u \]

\[ \left( \frac{\partial u}{\partial \tilde{t}} + u \cdot \nabla u \right) = -\nabla p + \frac{\mu}{\rho UL} \nabla^2 u. \quad \text{(A)} \]

\[ \frac{\rho UL}{\mu} \left( \frac{\partial u}{\partial \tilde{t}} + u \cdot \nabla u \right) = -\nabla q + \nabla^2 u. \quad \text{(B)} \]
Our solution depends upon only one nondimensional parameter.

(c) (3 points) A characteristic inertial force $F_i$ depends on the density of the fluid $\rho$, the characteristic velocity $U$, and the characteristic length $L$. Calculate $F_i$.

Since we are calculating a characteristic force (i.e., one by which you would divide the dimensional quantities in the problem), you may set any arbitrary constants equal to 1.

Solution. Matching units on the quantities, we see that the only way to “build” a force is to let

\[ F_i = \rho U^2 L^2 \]

so that

\[ [F_i] = [\rho U^2 L^2] = L^2 \left( \frac{M}{L^3} \right) \left( \frac{L^2}{T^2} \right) = ML T^2. \]

(d) (3 points) A characteristic viscous force $F_v$ depends on the bulk viscosity of the fluid $\mu$, the characteristic velocity $U$, and the characteristic length $L$. Calculate $F_v$.

Solution. Matching units on the quantities, we see that the only way to “build” a force is to let

\[ F_v = \mu U L. \]

(e) (2 points) The Reynolds number $Re$ of a system is given by the ratio of the inertial force to the viscous force. Rewrite your answer to (b) using the Re notation.

Solution. By our answers to (c) and (d), we have that

\[ Re = \frac{F_i}{F_v} = \frac{\rho U^2 L^2}{\mu U L} = \frac{\rho U L}{\mu}. \]

Therefore, equations (A) and (B) become

\[ \left( \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\nabla p + \frac{1}{Re} \nabla^2 \mathbf{u}. \]

\[ Re \left( \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\nabla q + \nabla^2 \mathbf{u}. \]

For any particular problem, we expect the dimensionless pressure to be $O(1)$. Therefore, we would use the $p$ equation when inertial forces dominate (and hence $Re$ is large) and we would use the $q$ equation when viscous forces dominate (and hence $Re$ is small).

2. Suppose that a person dips the end of a wooden spoon of radius $R$ into a viscous fluid (honey, for instance), then removes the spoon and holds it horizontal. The person then twirls the end of the spoon around with angular velocity $\Omega$ (see figure). The equation for the velocity $\tilde{u}$ of the fluid in the $\theta$-direction is given by

\[ \nu \frac{\partial^2 \tilde{u}}{\partial \tilde{z}^2} = g \cos \theta, \quad (1.1) \]
where $\nu$ is kinematic viscosity and $\tilde{z}$ is distance measured in the direction normal to the surface of the spoon. Here $\tilde{z} = 0$ corresponds to the surface of the spoon.

(a) (2 points) What is the velocity of the honey at the surface of the spoon?

*Solution.* The velocity of the honey at the surface of the spoon must be the same as the velocity at the surface of the spoon itself, which is $\Omega R$.

(b) (2 points) If $H$ is some characteristic height of the honey, find the characteristic acceleration due to the rotation of the spoon.

*Solution.* Since the acceleration due to gravity is listed on the right, the acceleration due to the rotation must be the quantity on the left-hand side of (1.1). If we use the characteristic velocity from (a), we have

$$\tilde{u}(\tilde{z}) = \Omega Ru(z), \quad \tilde{z} = Hz.$$  

Making these substitutions into (1.1), we obtain

$$\frac{\nu \Omega R}{H^2} \frac{\partial^2 u}{\partial z^2} = g \cos \theta,$$  

and hence a characteristic acceleration due to rotation must be $\nu \Omega R / H^2$.

(c) (2 points) Find the dimensionless parameter on which the solution must depend.

*Solution.* Continuing to simplify (C), we obtain

$$\alpha \frac{\partial^2 u}{\partial z^2} = \cos \theta, \quad \alpha = \frac{\nu \Omega R}{g H^2}.$$  

(D)  

Since (D) is dimensionless, the solution must depend only on $\alpha$.

(d) (4 points) How does the solution behave as $\Omega \to \infty$? How does it behave as $\Omega \to 0$?

*Solution.* As $\Omega \to \infty$, $\alpha \to \infty$. Therefore, we see that if we rewrite (D) as

$$\frac{\partial^2 u}{\partial z^2} = \alpha^{-1} \cos \theta,$$
we see that the right-hand side becomes zero and \( u \) must become nearly linear. In particular, the velocity will not depend on \( g \). This is because the spoon is spinning so fast that gravity will not pull the honey off.

Now consider the opposite regime. As \( \Omega \to 0 \), \( \alpha \to 0 \). This is the case where the spoon is stopped, and the honey flows off the spoon. Looking at our equations, we see that the velocity must become very large. This makes sense, since we scaled by the spoon velocity (which is going to zero), so any velocity of the honey would appear large. However, if \( \cos \theta = 0 \), then the fluid velocity becomes bounded (very near 0 in dimensional units). But \( \cos \theta = 0 \) correspond to the line of symmetry, with \( \theta = \pi/2 \) being the top of the spoon, which the honey flows away from, and \( \theta = -\pi/2 \) being the top of the spoon, where the honey is flowing just in the radial direction, with no angular velocity.

3. Suppose that in the spruce budworm model we replace the predation term by

\[
\tilde{P}(\tilde{N}) = \frac{B}{2} \left[ 1 + \tanh \left( \frac{\tilde{N} - A}{N_w} \right) \right],
\]

where \( N_w \ll A \) models the width of the transition region from no predation to full predation.

(a) (4 points) Use the same scalings as those given in class to scale the evolution equation that results.

**Solution.** The new equation is

\[
\frac{d\tilde{N}}{dt} = R\tilde{N} \left( 1 - \frac{\tilde{N}}{K} \right) - \frac{B}{2} \left[ 1 + \tanh \left( \frac{\tilde{N} - A}{N_w} \right) \right].
\]

Letting

\[
\tilde{N} = NA, \quad \tilde{t} = \frac{At}{B}, \quad R = \frac{Br}{A}, \quad K = Aq,
\]

we have

\[
B\dot{N} = RAN \left( 1 - \frac{N}{q} \right) - \frac{B}{2} \left[ 1 + \tanh \left( \frac{N - 1}{N_w/A} \right) \right]
\]

\[
\dot{N} = rN \left( 1 - \frac{N}{q} \right) - \frac{1}{2} \left[ 1 + \tanh \left( \frac{N - 1}{N_w/A} \right) \right]. \tag{E}
\]

Now let \( N_w/A = \epsilon \), where \( 0 < \epsilon \ll 1 \).

(b) (2 points) As we take the limit \( \epsilon \to 0 \), what values does our dimensionless predation term take on?

**Solution.** Our dimensionless predation term is now

\[
P(N) = \frac{1}{2} \left[ 1 + \tanh \left( \frac{N - 1}{\epsilon} \right) \right],
\]
which takes on the values

\[ P(N) = \begin{cases} 
0, & N < 1, \\
1, & N > 1, 
\end{cases} \quad \epsilon \to 0. \]

(c) (4 points) Above find a schematic of the \( q-r \) parameter plane which is divided into regions. In different regions, the number and/or (general) location of the steady states is different. Identify what is happening in each of the labeled regions, and verify that the curves listed are the appropriate boundaries of those regions.

**Solution.** To find the steady states, we must find the points where \( \dot{N} = 0 \). Using (E), we see that these points are given by

\[ rN \left( 1 - \frac{N}{q} \right) = \frac{1}{2} \left[ 1 + \tanh \left( \frac{N - 1}{\epsilon} \right) \right]. \quad (F) \]

(Refer to the figure, which shows examples from each region.) For the case where \( 1 - N \) is positive and \( O(1) \), the right-hand side of (F) vanishes and we have

\[ N \left( 1 - \frac{N}{q} \right) = 0, \]

which means that the steady states are \( N_* = 0, N_* = q \). But this second state is less than 1 only when \( q < 1 \). So when \( q < 1 \) (region I), we have only one positive steady state, which is less than 1. Next we consider the case when \( q > 1 \). For there to be a steady state near \( N = 1 \), the parabola’s value at 1 must be between 0 and 1, so

\[ 0 < r \left( 1 - \frac{1}{q} \right) < 1 \quad \implies \quad r < \frac{q}{q - 1}. \]
Therefore, if \( r > q/(q-1) \) and \( q > 1 \) (region II), there is no intersection near \( N = 1 \). Thus there is only one positive steady state with \( N_\ast > 1 \).

If (G) is satisfied, then there are either 1 or 3 positive steady states, depending on whether the parabola also intersects the upper part of the tanh. The maximum of the left-hand side occurs at \( N_\ast = q/2 \), and has the value \( rq/4 \). Therefore, for there to be three positive steady states, \( rq > 4 \) (so the maximum is high enough) and \( q > 2 \) (so that the upper branch of the tanh is intersected on both the up and down slopes). This is region IV. If either of these conditions are violated, we have only one positive steady state with \( N_\ast \approx 1 \) (region III).

In summary, \( N = 0 \) is always a steady state and:

- Region I: One positive steady state \( N_\ast < 1 \).
- Region II: One positive steady state \( N_\ast > 1 \).
- Region III: One positive steady state \( N_\ast \approx 1 \).
- Region IV: Three positive steady states: two greater than 1 and one approximately 1.

(d) (3 points) Classify each of the steady states as stable or unstable.

Solution. We know that the stability of the steady state is determined by \( f'(N_\ast) \), where \( N_\ast \) is the steady state and \( f \) is the right-hand side of (E). Therefore, from our graph we see that the roots alternate stability. In other words, zero is always unstable. The smallest positive steady state is stable. If there are three positive steady states, the largest positive steady states is stable, and the one between is unstable.