Homework Set 5 (Second Revision)

Read sections 9.1, 9.2.

Discrete Maps

1. Consider the discrete map

\[ x_{n+1} = x_n + \lambda \sin 2\pi x_n, \quad \lambda > 0, \quad 0 \leq x \leq 1. \]

(a) (3 points) Find the fixed points of the map and their stability for various values of \( \lambda \). Identify a critical value \( \lambda_c \) where all solutions become unstable.

(b) (7 points) Show that a stable two-cycle exists for \( \lambda \to \lambda_c^+ \). Hint: Let

\[ \lambda = \lambda_c + \frac{\delta^2}{2\pi}, \quad x = x_c + \frac{\delta x}{2\pi}, \quad 0 < \delta \ll 1. \]  

(5.1)

1-D Biological Discrete Models

2. Consider the discrete exponential model for insect populations given in class:

\[ N_{t+1} = N_t \exp(r(1 - N_t)). \]  

(a) (4 points) Show that if (5.2) has a 2-cycle, the periodic values must satisfy

\[ f_1(N) \equiv \frac{2}{N} - 1 = \exp(r(1 - N)) \equiv f_2(N), \quad N \neq 1. \]  

(5.3)

(b) (3 points) Show algebraically that (5.3) has exactly two solutions with \( N < 2 \) for all \( r > 2 \). (Hint: Consider each side of (5.3) in the neighborhood of \( N = 1 \).)

(c) (3 points) Using facts about the fixed points of the iteration, argue why the 2-cycle should be stable (at least for some values of \( r \) near 2). Do NOT attempt to use the criterion outlined in class.
Discrete Systems

3. Consider the system of difference equations

\[
\begin{align*}
x_{n+1} &= y_n, \\
y_{n+1} &= -x_n + 4y_n - 2y_n^2.
\end{align*}
\]

(a) (2 points) Find the fixed points.
(b) (3 points) Determine the stability of the origin.
(c) (5 points) Use the linearized form of the operator to explain why there should be orbits which are very close to 4-cycles in the neighborhood of the other fixed point.
(d) (3 points) Verify your answer to (c) using some sort of computational software.

Host-Parasitoid Systems

4. We consider the following case of a host-parasitoid system:

\[
\begin{align*}
1 - x_t &= \exp(-x_t y_t), \\
y_{t+1} &= \lambda y_t (1 - x_t), \quad \lambda > 0.
\end{align*}
\]

Here \(y_t\) is the population of the parasite, and \(x_t\) is the fraction of the host population that is infected. (In this system we treat the parasite as lethal; hence (5.5a) has no time evolution in it.)

(a) (5 points) Show that the origin is a fixed point, and find condition(s) on \(\lambda\) such that a second fixed point \((x_*, y_*)\) exists.
(b) (2 points) Show that for the origin to be stable, it is sufficient that \(0 < \lambda < 1\). Explain your result biologically.