The Error Function

The *error function* is defined by

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-z^2}\,dz.$$

It arises often in probability problems. Also, because of the relationship between the random walk and diffusion processes, it occurs often in diffusion problems.

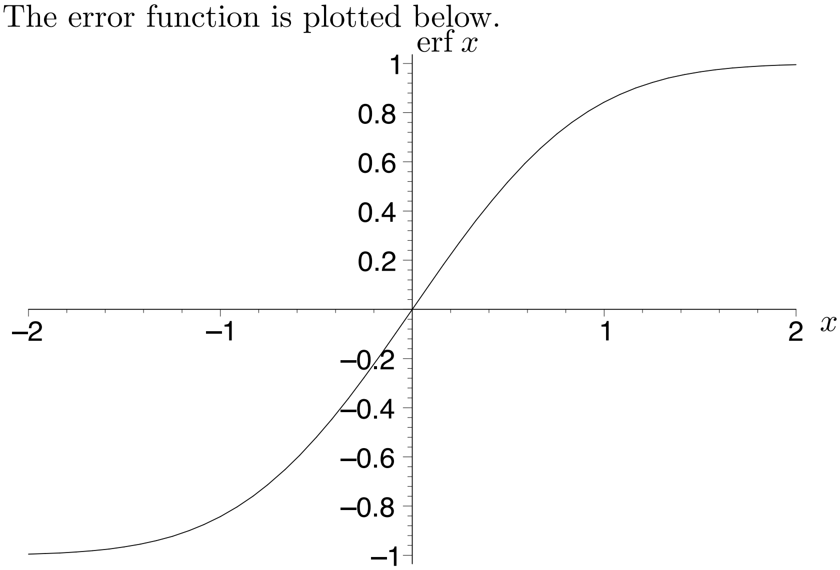
Since

$$\int_0^\infty e^{-z^2}\,dz = \frac{\sqrt{\pi}}{2}$$

(usually proven in MATH 243), the coefficient in front of the integral normalizes $\operatorname{erf}(\infty) = 1$. Moreover,

$$\operatorname{erf}(-x) = \frac{2}{\sqrt{\pi}} \int_0^{-x} e^{-z^2}\,dz = \frac{2}{\sqrt{\pi}} \int_0^{x} e^{-(-y)^2}\,(-dy) = -\operatorname{erf} x,$$

so $\operatorname{erf} x$ is odd. The error function is plotted below.



$\operatorname{erf} x$ *vs.* x.

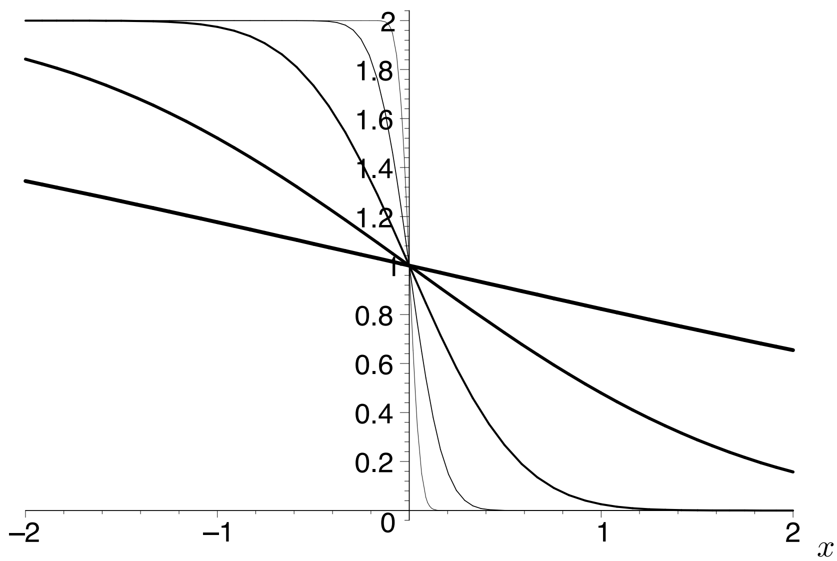
The *complementary error function* is defined by

$$\begin{aligned}\operatorname{erfc}(x) = 1 - \operatorname{erf}(x) &= \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-z^2}\, dz - \frac{2}{\sqrt{\pi}} \int_0^x e^{-z^2}\, dz \\ &= \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-z^2}\, dz.\end{aligned}$$

In the example in class, we found that the solution to a diffusion equation was given by

$$\operatorname{erfc}\left(\frac{x}{2\sqrt{t}}\right).$$

This function is plotted below for various values of t. Note that increasing t doesn't change the *shape* of the curve; it just changes the *scale*.



$\operatorname{erfc}(x/2\sqrt{t})$ *vs.* x. In increasing order of thickness: $t = 0.001$, 0.01, 0.1, 1, 10.