The Error Function

The *error function* is defined by

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_{0}^{x} e^{-z^2} \, dz.$$ 

It arises often in probability problems. Also, because of the relationship between the random walk and diffusion processes, it occurs often in diffusion problems.

Since

$$\int_{0}^{\infty} e^{-z^2} \, dz = \frac{\sqrt{\pi}}{2}$$

(usually proven in MATH 243), the coefficient in front of the integral normalizes $\text{erf}(\infty) = 1$. Moreover,

$$\text{erf}(-x) = \frac{2}{\sqrt{\pi}} \int_{0}^{-x} e^{-z^2} \, dz = \frac{2}{\sqrt{\pi}} \int_{0}^{x} e^{-(y)^2} (-dy) = -\text{erf} \, x,$$

so $\text{erf} \, x$ is odd. The error function is plotted below.

![Graph of erf(x) vs. x.](image)
The complementary error function is defined by
\[
\text{erfc}(x) = 1 - \text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_{0}^{\infty} e^{-z^2} \, dz - \frac{2}{\sqrt{\pi}} \int_{0}^{x} e^{-z^2} \, dz
\]
\[= \frac{2}{\sqrt{\pi}} \int_{x}^{\infty} e^{-z^2} \, dz.\]

In the example in class, we found that the solution to a diffusion equation was given by
\[\text{erfc}\left(\frac{x}{2\sqrt{t}}\right).\]
This function is plotted below for various values of \(t\). Note that increasing \(t\) doesn’t change the shape of the curve; it just changes the scale.

\[\text{erfc}(x/2\sqrt{t}) \text{ vs. } x. \text{ In increasing order of thickness: } t = 0.001, 0.01, 0.1, 1, 10.\]