Sine and Cosine Series (Revised)

We derived in class that the Fourier cosine series for the function

\[ f(x) = x^2 \cos x, \quad x \in (-\pi, \pi) \]

is given by

\[ f(x) = -2 + \left( \frac{1}{2} + \frac{\pi^2}{3} \right) \cos x + 2 \sum_{n=2}^{\infty} \left[ \frac{(-1)^{n+1}}{(n+1)^2} + \frac{(-1)^{n-1}}{(n-1)^2} \right] \cos nx. \]

Note that \( f(x) \) is smooth in its domain, but \( f' \) is discontinuous at \( x = \pm \pi \).

Below are plotted the extension of the function \( f \) to the domain \( x \in (-3\pi, 3\pi) \) (thickest line), as well as the Fourier series taking the first three, four, and five terms in the sum.

In increasing order of thickness: Fourier series keeping the first three, four, and five terms in the series, as well as \( f(x) \) vs. \( x \).
The Fourier sine series $f_s$ for the odd extension of the function $f(x) = x(1 - x)$, $x \in [0, 1]$ to the region $x \in [-1, 1]$ is given by

$$f_s(x) = \frac{4}{\pi^3} \sum_{n=1}^{\infty} \frac{1 - (-1)^n}{n^3} \sin n\pi x.$$ 

In increasing order of thickness: $f_s$ keeping the first one, two, and three nonzero terms in the series, as well as the odd extension of $f(x)$ vs. $x$.

**Remarks**

1. The terms decay quickly (like $n^{-3}$).
2. The Fourier coefficient is zero if $n$ is even.
The Fourier cosine series $f_c$ for the even extension of the function $f(x) = x(1 - x)$, $x \in [0, 1]$ to the region $x \in [-1, 1]$ is given by

$$f_c(x) = \frac{1}{6} - \frac{2}{\pi^2} \sum_{n=2}^{\infty} \frac{(-1)^n + 1}{n^2} \cos n\pi x.$$ 

In increasing order of thickness: $f_c$ keeping the first two, three, and four nonzero terms in the series, as well as the even extension of $f(x)$ vs. $x$.

**Remarks**

1. The terms do not decay as quickly (like $n^{-2}$). As a result, you need more of them to get the same accuracy as the sine series.
2. The series solutions are most inaccurate where the solution has a kink.
3. The Fourier coefficient is zero if $n$ is odd.