# Legendre Polynomials

If $n$ is an integer, then there exists a polynomial solution to the Legendre equation of order $n$:

$$(1-z^2)\frac{d^2y}{dz^2} - 2z\frac{dy}{dz} + n(n+1)y = 0, \qquad z \in [-1,1].$$

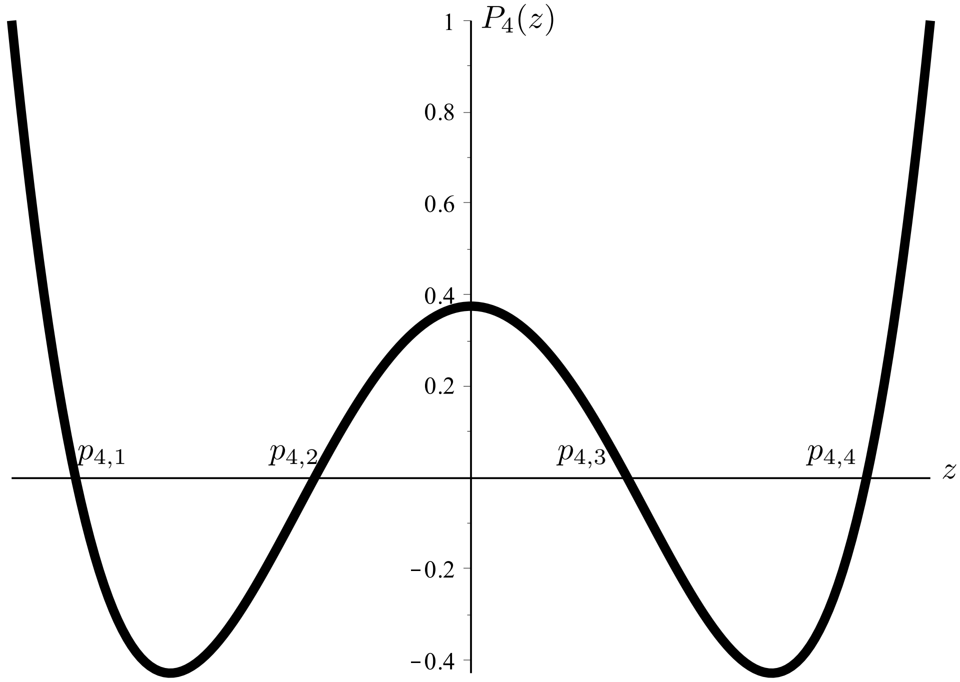
This solution is called the *Legendre polynomial* $P_n(z)$. $z$ is taken in this range because most often, $z = \cos\phi$, where $\phi$ is the *colatitude* (angle measured from the north pole) in spherical coordinates.

In deriving results about the Legendre polynomials, the following equations proved useful:

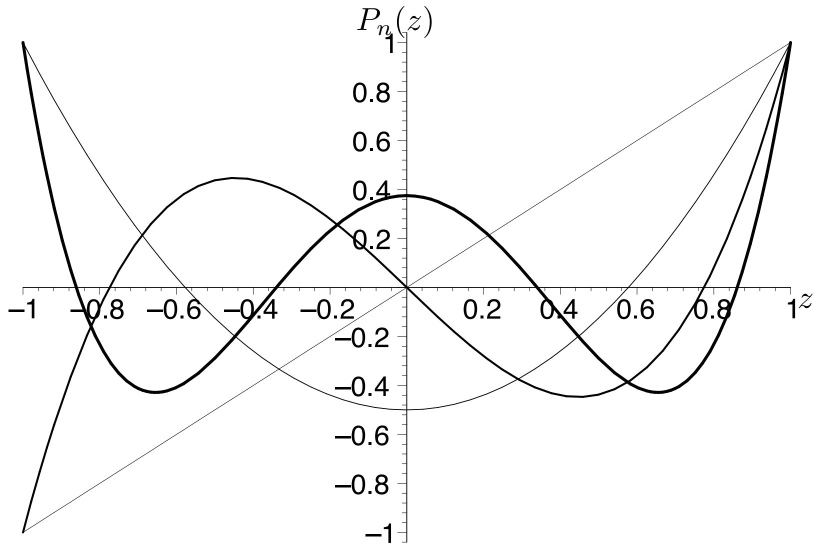
$$\frac{d^j[zf(z)]}{dz^j} = zf^{(j)} + jf^{(j-1)}, \tag{1}$$

$$\frac{d^j[(1-z^2)g(z)]}{dz^j} = (1-z^2)g^{(j)} - 2jzg^{(j-1)} - j(j-1)g^{(j-2)}. \tag{2}$$

The $i$th zero of $P_n(z)$ (measured from left to right) is denoted $p_{n,i}$.




The four zeroes of $P_4(z)$.




$P_n(z)$ *vs.* $z$ for $n = 1, 2, 3, 4$ (in increasing order of thickness).

Here is a plot of $P_n(z)$ for various $n$. Note that:

1. $P_n$ is odd if $n$ is odd and even if $n$ is even.
2. $P_n(1) = 1$.
3. $|P_n(z)| \leq 1$.