MATH 535-011 Introduction to Partial Differential Equations
Prof. D. A. Edwards DDue: Apr. 29, 2015

Updates

1. Exam II will be administered on Wednesday, Apr. 22. You will need to bring a **SMALL** blue book. It will cover up through Poisson's Integral Formula, so you should do the first three homework problems before the exam for practice.

Homework Set 7 (Second Revision)

Read sections 1.5, 2.5.1, 2.5.2, 2.5.4, 7.10.1–7.10.3.

Laplace's Equation

1. (7 points) Consider Laplace's equation inside a rectangle $0 \le x \le 1$, $0 \le y \le \pi$, with the boundary conditions

$$\frac{\partial u}{\partial x}(0,y) = f(y), \qquad \frac{\partial u}{\partial y}(x,0) = \frac{\partial u}{\partial y}(x,\pi) = 0, \qquad \frac{\partial u}{\partial x}(1,y) = g(y). \tag{7.1}$$

What is the solvability condition and its physical interpretation? (*Hint: Be very careful about the outward-pointing normals, and the treatment of dx and dy in the path integral.*)

Poisson's Equation

2. Suppose we want to solve Poisson's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = xh(y); \qquad 0 \le x \le 1, \quad 0 \le y \le \pi; \qquad \int_0^\pi h(y) = 0, \tag{7.2a}$$

subject to the inhomogeneous boundary conditions in (7.1) with

$$f(y) = g(y) = 1. \tag{7.2b}$$

(a) (2 points) Verify that (7.2b) satisfies your answer to #2.
(b) (14 points) Show that the solution to (7.2) is given by

$$u(x,y) = x + c_2 + \sum_{n=1}^{\infty} \frac{h_n \cos ny}{n^2} \left[\frac{\cosh nx - \cosh n(x-1)}{n \sinh n} - x \right],$$

and find the value of h_n. You should find that c_2 is arbitrary: why should this be?

Poisson's Integral Formula

3. Consider the following Dirichlet problem for Laplace's equation:

$$\frac{\partial}{\partial r}\left(r\frac{\partial \psi}{\partial r}\right) + \frac{1}{r}\frac{\partial^2 \psi}{\partial \theta^2} = 0, \quad r \le 1; \qquad \psi(1,\theta) = 1, \quad \psi(0,\theta) \text{ bounded.}$$

(a) (3 points) Find the solution to this problem by inspection.
(b) (7 points) Use your answer to (a) and Poisson's integral formula to determine that

$$\int_0^{2\pi} \frac{d\phi}{A - B\cos\phi} = \frac{2\pi}{\sqrt{A^2 - B^2}}; \qquad A > 0, \quad B < A. \tag{7.3}$$

Spherical Harmonics

4. Consider Laplace's equation in spherical coordinates:

$$\frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{\sin\phi}\frac{\partial}{\partial \phi}\left(\sin\phi\frac{\partial u}{\partial \phi}\right) + \frac{1}{\sin^2\phi}\frac{\partial^2 u}{\partial^2 \theta}\right] = 0,$$

For each of the domains and boundary conditions given below, write down the appropriate r- and θ-dependent eigenfunctions. Also, write down the equation for $\Phi(z)$ that results. (Do not attempt to solve the Φ equation.) You do not have to repeat any analysis in the class notes.

(a) (3 points) $0 < \theta < 2\pi$, $0 < \phi < \pi$, $r > 1$
(b) (4 points)

$$0 < \theta < \pi/2, \quad 0 < \phi < \pi, \quad 0 < r < 1; \qquad \frac{\partial u}{\partial \theta}(r, 0, \phi) = u(r, \pi/2, \phi) = 0. \tag{7.4}$$