Classification

Consider the following partial differential equation:

$$A\frac{\partial^2\phi}{\partial x^2} + B\frac{\partial^2\phi}{\partial x\partial y} + C\frac{\partial^2\phi}{\partial y^2} + D\frac{\partial\phi}{\partial x} + E\frac{\partial\phi}{\partial y} + F\phi + G = 0, \tag{1}$$

where all the capital letters represent functions of x and y. It has characteristics defined by

$$\frac{dy}{dx} = \frac{B \pm \sqrt{B^2 - 4AC}}{2A}. \tag{2}$$

To classify the equation and write it in canonical form, we introduce new variables x and y and rewrite (1) as

$$\mathcal{F}[\xi]\frac{\partial^2\phi}{\partial\xi^2} + \mathcal{G}[\xi,\eta]\frac{\partial^2\phi}{\partial\xi\partial\eta} + \mathcal{F}[\eta]\frac{\partial^2\phi}{\partial\eta^2} + \text{lower order terms} = 0, \tag{3}$$

where

$$\mathcal{F}[\xi] = A\left(\frac{\partial\xi}{\partial x}\right)^2 + B\left(\frac{\partial\xi}{\partial x}\right)\left(\frac{\partial\xi}{\partial y}\right) + C\left(\frac{\partial\xi}{\partial y}\right)^2,$$
$$\mathcal{G}[\xi,\eta] = 2A\left(\frac{\partial\xi}{\partial x}\right)\left(\frac{\partial\eta}{\partial x}\right) + 2C\left(\frac{\partial\xi}{\partial y}\right)\left(\frac{\partial\eta}{\partial y}\right) + B\left[\left(\frac{\partial\xi}{\partial x}\right)\left(\frac{\partial\eta}{\partial y}\right) + \left(\frac{\partial\xi}{\partial y}\right)\left(\frac{\partial\eta}{\partial x}\right)\right].$$

Here $\mathcal{F}[\xi] = 0$ if $\xi =$ constant describes a characteristic. It can be shown that

$$\operatorname{sgn}(B^2 - 4AC) = \operatorname{sgn}(\mathcal{G}[\xi,\eta]^2 - 4\mathcal{F}[\xi]\mathcal{F}[\eta]). \tag{4}$$

Hyperbolic Case

In the hyperbolic case, $B^2 > 4AC$ and there are two real sets of characteristics given by (2). So we can set $\mathcal{F}[\xi] = \mathcal{F}[\eta] = 0$ and divide (3) by $\mathcal{G}[\xi,\eta]$ to obtain

$$\frac{\partial^2\phi}{\partial\xi\partial\eta} + \text{lower order terms} = 0,$$

which is the canonical form for the hyperbolic case.

Parabolic Case

In the parabolic case, $B^2 = 4AC$ and there is only one real set of characteristics given by (2). So we can set $\mathcal{F}[\xi] = 0$, and by (4) we have $\mathcal{G}[\xi, \eta] = 0$. Hence we may divide (3) by $\mathcal{F}[\eta]$ to obtain

$$\frac{\partial^2 \phi}{\partial \eta^2} + \text{lower order terms} = 0,$$

which is the canonical form for the hyperbolic case.

Elliptic Case

In the elliptic case, $B^2 < 4AC$ and there are no real characteristics. However, we still have two sets of coordinates at our disposal. Hence we choose coordinates so that $\mathcal{G}[\xi, \eta] = 0$ and $\mathcal{F}[\xi] = \mathcal{F}[\eta]$. Hence we may divide (3) by the $\mathcal{F}$ terms to obtain

$$\frac{\partial^2 \phi}{\partial \xi^2} + \frac{\partial^2 \phi}{\partial \eta^2} + \text{lower order terms} = 0,$$

which is the canonical form for the elliptic case.

To do this, we define ζ-curves *via*

$$\frac{dy}{dx} = \frac{2C}{B},$$

which will zero out the $\mathcal{G}$ term and yield

$$A' \frac{\partial^2 \phi}{\partial \zeta^2} + C' \frac{\partial^2 \phi}{\partial y^2} + \text{lower order terms} = 0,$$

and then define ξ and η *via*

$$\sqrt{A'} \frac{\partial \xi}{\partial \zeta} = \sqrt{C'} \frac{\partial \eta}{\partial y}, \qquad \sqrt{C'} \frac{\partial \xi}{\partial y} = -\sqrt{A'} \frac{\partial \eta}{\partial \zeta}.$$