Shadow Variables

Consider the following system:

\[ \begin{align*}
3x_1 + 2x_2 & \leq \beta_1 \\
4x_1 + x_2 & \leq \beta_2, \\
x_1 - x_2 &= \beta_3
\end{align*} \]

\[ \mathbf{x} \geq \mathbf{0}, \quad \max U(\mathbf{x}) = x_1 + x_2. \]

We begin by taking the case of

\[ \bar{\beta}_x = \begin{pmatrix} 5 \\ 5.06 \\ 0 \end{pmatrix} \implies \mathbf{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad U(\mathbf{x}) = 2. \]

Graphs are shown below. In each case, the shaded region is the feasible region for the two inequalities, and the circle indicates the intersection of the two boundaries. Hence the thick solid line represents the feasible region for the system. The dotted lines are lines of constant \( U \), with the optimal value indicated with the thickest line.

Case 1. Note that that the less steep line (first constraint) determines the feasible region.
Next, we take

\[ \tilde{\beta}_z = \begin{pmatrix} 4.9 \\ 5.06 \\ 0 \end{pmatrix} \implies z = \begin{pmatrix} 0.98 \\ 0.98 \end{pmatrix}, \quad U(z) = 1.96. \]

The dual of the simplified algebraic problem is given by

\[ 5y_1 + y_2 \geq 2, \quad y \geq 0, \quad \min V(y) = \tilde{\beta}^T y, \]

which has solution

\[ y = \begin{pmatrix} 0.4 \\ 0 \end{pmatrix} \]

for both \( \tilde{\beta}_x \) and \( \tilde{\beta}_z \). Hence this case follows the shadow variable formula, since the first constraint still determines the feasible region:

\[ U(z) = U(x) + y^T \begin{pmatrix} -0.1 \\ 0 \end{pmatrix} = 2 - 0.04 = 1.96. \]

Case 2. Note that that the less steep line (first constraint) determines the feasible region.
Lastly, we take

\[ \vec{\beta}_Z = \begin{pmatrix} 5.1 \\ 5.06 \\ 0 \end{pmatrix} \implies Z = \begin{pmatrix} 1.012 \\ 1.012 \end{pmatrix}, \quad U(Z) = 2.024. \]

In this case the shadow variable formula fails, since it is now the second constraint that determines the feasible region, so we have a different set of basic variables:

\[ U(Z) = 2.024 \neq U(x) + y^T \begin{pmatrix} 0.1 \\ 0 \end{pmatrix} = 2 + 0.04 = 2.04. \]

Case 3. Note that that the steeper line (second constraint) determines the feasible region.