Homework Set 9 Solutions

1. Consider the standard maximum problem

\[ \begin{align*}
A \mathbf{x} & \leq (1 - \lambda) \mathbf{b}_1 + \lambda \mathbf{b}_2, \quad \lambda \in [0, 1]; \\
A & \in \mathbb{R}^{m \times n}, \quad \mathbf{x} \in \mathbb{R}^n, \quad \mathbf{b}_i \in \mathbb{R}^m, \quad (9.1a) \\
\mathbf{x} & \geq 0, \quad \max_{\mathbf{x}} U(\mathbf{x}; \lambda), \quad U(\mathbf{x}; \lambda) = \mathbf{c}^T \mathbf{x}; \quad \mathbf{c} \in \mathbb{R}^n. \quad (9.1b)
\end{align*} \]

(a) (4 points) Prove that if (9.1) has a feasible solution for \( \lambda = 0 \) and \( \lambda = 1 \), it has a feasible solution for \( \lambda \in [0, 1] \).

Solution. Let

\[ A \mathbf{x}_1 \leq \mathbf{b}_1, \quad A \mathbf{x}_2 \leq \mathbf{b}_2, \]

so \( \mathbf{x}_j \) is a feasible solution to the corresponding maximum problem for \( \lambda = 0 \) and \( \lambda = 1 \). But then

\[ A((1 - \lambda)\mathbf{x}_1 + \lambda \mathbf{x}_2) = (1 - \lambda)A\mathbf{x}_1 + \lambda A\mathbf{x}_2 \leq (1 - \lambda)\mathbf{b}_1 + \lambda \mathbf{b}_2, \]

so \( \mathbf{x} = (1 - \lambda)\mathbf{x}_1 + \lambda \mathbf{x}_2 \) is a feasible solution for \( \lambda \in [0, 1] \). (The fact that \( \mathbf{x} \geq \mathbf{0} \) has been established previously.)

(b) (4 points) Use your answer to (a) and the dual problem to prove that if (9.1) has an optimal solution for \( \lambda = 0 \) and \( \lambda = 1 \), it has an optimal solution for \( \lambda \in [0, 1] \).

Solution. Now consider the dual problem. Since there is an optimal solution for the primal problem for \( \lambda = 0 \) and \( \lambda = 1 \), there must be a feasible problem for the dual problem as well. But the constraint part of the dual problem is \( A^T \mathbf{y} \geq \mathbf{c} \), which is independent of \( \lambda \). So there must be a feasible solution to the dual problem for all \( \lambda \in [0, 1] \). Then by the duality theorem, since there are feasible solutions to both the feasible and dual problems, there must be an optimal solution to both the feasible and dual problems for \( \lambda \in [0, 1] \).

Now we replace the inequalities with equalities and consider the case

\[ \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \mathbf{x} = \mathbf{b}, \quad \mathbf{b} = (1 - \lambda) \begin{pmatrix} 1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 9 \end{pmatrix}, \quad \mathbf{c} = -\begin{pmatrix} 3 \\ 3 \\ 5 \end{pmatrix}. \quad (9.2) \]

(c) (8 points) Find the optimal solution \( \mathbf{x} \) as a function of \( \lambda \in [0, 1] \), and graph

\[ \max_{\mathbf{x}} U(\mathbf{x}; \lambda) \]

as a function of \( \lambda \). What happens when the optimal solution changes from one extreme point to another?
Solution. Since we have equalities, we can check the basic solutions:

\[
\begin{pmatrix}
2 & 3 \\
5 & 6
\end{pmatrix}
\begin{pmatrix}
x_2 \\
x_3
\end{pmatrix}
= \begin{pmatrix}
1 + 3\lambda \\
3 + 6\lambda
\end{pmatrix}
\]

\[
\begin{pmatrix}
x_2 \\
x_3
\end{pmatrix}
= -\frac{1}{3}
\begin{pmatrix}
6 & -3 \\
-5 & 2
\end{pmatrix}
\begin{pmatrix}
1 + 3\lambda \\
3 + 6\lambda
\end{pmatrix}
= \frac{1}{3}
\begin{pmatrix}
3 \\
-1 + 3\lambda
\end{pmatrix},
\]

\[
z_1 = \begin{pmatrix}
0 \\
\frac{1}{\lambda - 1/3}
\end{pmatrix},
\]

\[
U(z_1; \lambda) = -3 - 5\left(\lambda - \frac{1}{3}\right) = -5\lambda - \frac{4}{3},
\]

\[
\begin{pmatrix}
1 & 3 \\
4 & 6
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_3
\end{pmatrix}
= \begin{pmatrix}
1 + 3\lambda \\
3 + 6\lambda
\end{pmatrix}
\]

\[
\begin{pmatrix}
x_1 \\
x_3
\end{pmatrix}
= -\frac{1}{6}
\begin{pmatrix}
6 & -3 \\
-4 & 1
\end{pmatrix}
\begin{pmatrix}
1 + 3\lambda \\
3 + 6\lambda
\end{pmatrix}
= \frac{1}{6}
\begin{pmatrix}
3 \\
1 + 6\lambda
\end{pmatrix},
\]

\[
z_2 = \begin{pmatrix}
1/2 \\
0
\end{pmatrix},
\]

\[
U(z_2; \lambda) = -3 - \frac{1}{2} - 5\left(\lambda + \frac{1}{6}\right) = -5\lambda - \frac{14}{6} = -5\lambda - \frac{7}{3},
\]

\[
\begin{pmatrix}
1 & 2 \\
4 & 5
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix}
= \begin{pmatrix}
1 + 3\lambda \\
3 + 6\lambda
\end{pmatrix}
\]

\[
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix}
= -\frac{1}{3}
\begin{pmatrix}
5 & -2 \\
-4 & 1
\end{pmatrix}
\begin{pmatrix}
1 + 3\lambda \\
3 + 6\lambda
\end{pmatrix}
= \frac{1}{3}
\begin{pmatrix}
1 - 3\lambda \\
1 + 6\lambda
\end{pmatrix},
\]

\[
z_3 = \begin{pmatrix}
-\frac{\lambda + 1/3}{2} \\
0
\end{pmatrix},
\]

\[
U(z_3; \lambda) = -3\left(-\lambda + \frac{1}{3}\right) - 3\left(2\lambda + \frac{1}{3}\right) = -3\lambda - 2.
\]

First we note that \(z_1\) is a feasible solution only for \(\lambda \in [1/3, 1]\), and \(z_3\) is a feasible solution only for \(\lambda \in [0, 1/3]\). Therefore, for \(\lambda \in [1/3, 1]\), \(z_1\) is optimal because trivially \(U(z_1; \lambda) > U(z_2; \lambda)\). For \(\lambda \in [0, 1/3]\), we solve to find where

\[
U(z_2; \lambda) = U(z_3; \lambda)
\]

\[-5\lambda - \frac{7}{3} = -3\lambda - 2
\]

\[2\lambda = -\frac{1}{3}.
\]

Hence one of these two functions is always larger than the other. Since \(U(z_3; 0) = -2 > -7/3 = U(z_2; 0)\), we see that \(U(z_2; \lambda)\) is always smaller. Hence we have that

\[
\max_{x} U(x; \lambda) = \begin{cases} 
U(z_3; \lambda) = -3\lambda - 2, & \lambda \in [0, 1/3], \\
U(z_1; \lambda) = -5\lambda - \frac{4}{3}, & \lambda \in [1/3, 1].
\end{cases}
\]

The graph is shown below. We note that at \(\lambda = 1/3\), where the optimal solution changes from \(z_1\) to \(z_3\), \(z_1 = z_3 = (0, 1, 0)\).
Figure 9.1. Solution for part (c). Dotted line indicates the point \((1/3, -3)\) where the optimal solution changes. Left line: \(U(z_3)\). Right line: \(U(z_1)\).

(d) (6 points) Repeat the analysis of part (c) if

\[
b = \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \quad c = -\begin{pmatrix} 3 \\ 3 \\ 4 - 2\lambda \end{pmatrix}, \quad \lambda \in [0, 1].
\]

In particular, show that there are an infinite number of optimal solutions for a particular value of \(\lambda\).

**Solution.** \(b\) in this case is the same as \(b\) in the previous case with \(\lambda = 0\); therefore, we have that the two feasible solutions are

\[
z_2 = \begin{pmatrix} 1/2 \\ 0 \\ 1/6 \end{pmatrix}, \quad U(z_2; \lambda) = -3 \cdot \frac{1}{2} - (4 - 2\lambda) \frac{1}{6} = \frac{\lambda}{3} - \frac{13}{6},
\]

\[
z_3 = \begin{pmatrix} 1/3 \\ 1/3 \\ 0 \end{pmatrix}, \quad U(z_3; \lambda) = -3 \cdot \frac{1}{3} - 3 \cdot \frac{1}{3} = -2.
\]

Setting these two expressions equal, we have

\[
\frac{\lambda}{3} - \frac{13}{6} = -2 \quad \Rightarrow \quad \lambda = \frac{1}{2}.
\]
Plugging in $\lambda = 0$, we have that $U(z_3; 0) > U(z_2; 0)$. Therefore, we see that for $\lambda \in [0, 1/2]$, $z_3$ is optimal. For $\lambda \in [1/2, 1]$, $z_2$ is optimal. Hence we have that

$$\max_x U(x; \lambda) = \begin{cases} 
-2, & \lambda \in [0, 1/2], \\
\frac{\lambda}{3} - \frac{13}{6}, & \lambda \in [1/2, 1].
\end{cases}$$

The graph is shown below.

![Graph showing solution for part (d).](image)

Figure 9.2. Solution for part (d). Dotted line: $U(z_3)$. Dashed line: $U(z_2)$. Solid line: maximum.

By Homework Set 8, #3(b), we know that every convex combination of basic optimal solutions is an optimal solution. For $\lambda = 1/2$, we have that $z_2$ and $z_3$ are both basic optimal solutions, and hence $x = (1 - \alpha)z_2 + \alpha z_3$ is also an optimal solution for any $\alpha \in [0, 1]$.

2. Now let's consider the case where the matrix in the linear programming problem is perturbed. Therefore, we replace the regular matrix $A$ in our problem with

$$A(\epsilon) = A(0) + \epsilon P, \quad A, P \in \mathbb{R}^{n \times n}. \tag{9.3a}$$

Since $A$ depends on $\epsilon$, our solution will as well. Consider the particular example

$$A(\epsilon)x(\epsilon) = b, \quad x(\epsilon) \geq 0, \quad f(\epsilon) = \min_{x(\epsilon)} c^T x(\epsilon); \quad x(\epsilon), b, c \in \mathbb{R}^n. \tag{9.3b}$$

(a) (4 points) Assume that $A(\epsilon)$ is invertible for any $\epsilon \neq 0$, $M(\epsilon) = [A(\epsilon)]^{-1}$. Show that

$$\frac{dM}{d\epsilon} = -M(\epsilon)PM(\epsilon). \tag{9.4}$$
Solution. Dropping the $\epsilon$ argument for simplicity, we have

$$
\frac{dM}{d\epsilon} A + M \frac{dA}{d\epsilon} = 0
$$

$$
\frac{dM}{d\epsilon} = (-MP)A^{-1}
$$

$$
= -MPM,
$$

where we have used the fact that $\frac{dA}{d\epsilon} = P$.

Now consider the following problem:

$$
(1 - 2\epsilon)x_1 - (1 + 2\epsilon)^{-1}x_2 = -2, \quad 0 \leq \epsilon \ll 1,
$$

$$
x_1 - x_2 = -2,
$$

$$
x \geq 0, \quad \min_x x_2.
$$

(9.5)

(b) (4 points) Solve (9.5) for $\epsilon \neq 0$.

Solution. We note that $x_2 = x_1 + 2$ from the second equation in (9.5), so the first equation becomes

$$
(1 - 2\epsilon)x_1 - (1 + 2\epsilon)^{-1}(x_1 + 2) = -2
$$

$$
(1 - 2\epsilon)(1 + 2\epsilon)x_1 - (x_1 + 2) = -2(1 + 2\epsilon)
$$

$$
x_1[(1 - 4\epsilon^2) - 1] = -4\epsilon
$$

$$
x_1 = \epsilon^{-1}, \quad x_2 = 2 + \epsilon^{-1},
$$

so the minimum cost is $2 + \epsilon^{-1}$.

(c) (2 points) Solve (9.5) for $\epsilon = 0$.

Solution. Substituting $\epsilon = 0$ into the first equation of (9.5), we have

$$
x_1 - x_2 = -2,
$$

which is the second equation in (9.5). Hence $x_2 = x_1 + 2$, $x_1 \geq 0$, which means that the minimum cost is 2.

(d) (3 points) Do your answers agree with each other in the limit that $\epsilon \to 0$? Explain why using the notation of (9.3).

Solution. They do not agree, because the minimum cost goes to $\infty$ in part (a) as $\epsilon \to 0$, while the minimum cost in part (b) is 1. This is because for $\epsilon \neq 0$, $M(\epsilon)$ exists and we have a unique solution, while for $\epsilon = 0$, $A(0)$ is not invertible.

(e) (5 points) Use your results to give conditions on the system under which

$$
f(0) = \lim_{\epsilon \to 0} f(\epsilon),
$$

(9.6)
and prove that those conditions are sufficient.

**Solution.** Given the discussion above, we hypothesize that (9.6) holds if $A(0)$ is invertible. Calculating the derivative, we have

$$\frac{df}{d\epsilon} = \nabla f \cdot \frac{dx}{d\epsilon} = c^T \frac{dM}{d\epsilon} b = -c^T MPM b,$$

where we have used (9.4). Therefore, if $A(0)$ is invertible, $M(0)$ exists, as does $f'(0)$. And since a differentiable function is continuous, we are done.