Homework Set 8 Solutions

1. (4 points) In class, we listed the following two definitions of an extreme point \( z \) of a convex set \( X \):

   (a) \( z \) is not a convex combination of two other points in \( X \).
   
   (b) there do not exist two distinct points \( x, y \in X \) such that \( z = (1 - \lambda)x + \lambda y \), \( \lambda \in (0, 1) \).

   Prove that these two definitions are equivalent.

   **Solution.** We prove instead that the negative of the definitions are equivalent. Suppose (b) is false, and there exist two distinct points \( x, y \in X \) such that \( z = (1 - \lambda)x + \lambda y \), \( \lambda \in (0, 1) \). Since \( \lambda \neq 0, \lambda \neq 1 \), we see that \( z \neq x, z \neq y \). Therefore, \( z \) is a convex combination of two other points in \( X \), so (a) is false. Similarly, suppose (a) is false, and let \( z \) is a convex combination of two other points in \( X \), namely \( x \) and \( y \). Then \( z = (1 - \lambda)x + \lambda y, \lambda \in [0, 1] \). But if \( \lambda = 0, z = x \), which is a contradiction. Similarly, if \( \lambda = 1, z = y \), which is a contradiction. Therefore, there exist two distinct points \( x, y \in X \) such that \( z = (1 - \lambda)x + \lambda y, \lambda \in (0, 1) \), so (b) is false.

2. (2 points per part) Let \( S_1 \) and \( S_2 \) be disjoint, bounded, closed, convex regions in \( \mathbb{R}^2 \), \( N_j \) be the number of extreme points of \( S_j \), \( N_1 \geq N_2 \), \( N_2 \) finite. Let \( S' \) be the convex hull of \( S_1 \) and \( S_2 \), \( N' \) be the number of extreme points of \( S' \). Draw figures illustrating these cases. You do not need to go into a lengthy proof as long as your figures are clear.

   (a) \( N_1 < N' < \infty \)
   
   (b) \( N_1 < N' = \infty \)
   
   (c) \( N' < N_1 < \infty \)
   
   (d) \( N' < N_1 = \infty \)

   **Solution.**

   Figure 8.1. Dark region: \( S_1 \). Light region: \( S_2 \). Entire enclosed region: \( S' \).
See the figures above. In the first figure, \( S_1 \) has three extreme points (the corners of the triangle) while \( S' \) has four (the corners of the rectangle). In the second figure, \( S_1 \) has infinitely many extreme points (the arc, by analogy with the discussion in class about \( B_\delta(x_0) \)), but \( S' \) has the entire arc as extreme points, as well as the top corner. In the third, \( S_1 \) has five extreme points (the corners of the pentagon) while \( S' \) has four (the corners of the rectangle). In the last, \( S_1 \) has infinitely many extreme points, since each of the points on the top arc is an extreme point. However, \( S' \) has just five extreme points (the corners of the pentagon).

3. Consider the maximum problem in canonical form:

\[
\max c^T x, \quad A x = b, \quad A \in \mathbb{R}^{m \times n}, \quad c \in \mathbb{R}^n, \quad x \geq 0 \in \mathbb{R}^n, \quad b \in \mathbb{R}^m.
\]  

(8.1)

(a) (2 points) Prove that any convex combination of feasible basic solutions to (8.1) is also feasible.

Solution. Let \( x_1 \) and \( x_2 \) be feasible basic solutions of (8.1), so they are non-negative and satisfy \( A x_i = b \). Moreover, let \( y = (1 - \lambda)x_1 + \lambda x_2, \lambda \in [0,1] \) be their convex combination. Then \( y \geq 0 \) by previous work, and

\[
A y = A[(1 - \lambda)x_1 + \lambda x_2] = (1 - \lambda)A x_1 + \lambda A x_2 = (1 - \lambda) b + \lambda b,
\]

where we have used the fact that the \( x_i \) are feasible. But the last sum is just \( b \), so \( y \) is feasible.

(b) (2 points) Prove that any convex combination of optimal basic solutions to (8.1) is also optimal.

Solution. Let \( x_1 \) and \( x_2 \) be optimal basic solutions, so they satisfy all the restrictions of part (a), and also

\[
\max c^T x = c^T x_i \equiv \alpha.
\]

Then defining \( y \) as in part (a),

\[
c^T y = c^T[(1 - \lambda)x_1 + \lambda x_2] = (1 - \lambda)c^T x_1 + \lambda c^T x_2 = (1 - \lambda)\alpha + \lambda \alpha,
\]

where we have used (A). But the last sum is \( \alpha \), which is the maximum, so \( y \) is also optimal.

4. Consider the following problem:

\[
\begin{pmatrix} 2 & 0 & 3 \\ 1 & 1 & 1 \end{pmatrix} x = \begin{pmatrix} 12 \\ 5 \end{pmatrix}, \quad x \geq 0.
\]

(a) (4 points) Find all basic feasible solutions to the problem.

Solution. Zeroing out one column at a time, we have

\[
\begin{pmatrix} 0 & 3 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 12 \\ 5 \end{pmatrix} \quad \Rightarrow \quad 3x_3 = 12 \quad \Rightarrow \quad x_2 + x_3 = 5 \quad \Rightarrow \quad z_1 = \begin{pmatrix} 0 \\ 1/4 \end{pmatrix},
\]
\[
\begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_3 \end{pmatrix} = \begin{pmatrix} 12 \\ 5 \end{pmatrix} \implies 2x_1 + 3x_3 = 12 \implies z_2 = \begin{pmatrix} 3 \\ 2 \end{pmatrix},
\]

\[
\begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 12 \\ 5 \end{pmatrix} \implies 2x_1 = 12 \implies z_3 = \begin{pmatrix} 6 \\ -1 \\ 0 \end{pmatrix}.
\]

Therefore, \( z_3 \) is not feasible, so the desired result is \( z_1 \) and \( z_2 \).

(b) (2 points) Find the optimal basic solution if we wish to maximize \( U(x) = 3x_1 - 2x_2 + x_3 \).

Solution.

\[
U(z_1) = 3(0) - 2(1) + 4 = 2 \]

\[
U(z_2) = 3(3) - 2(0) + 2 = 11,
\]

so \( z_2 \) is optimal.

5. Consider the standard maximum problem

\[
\begin{align*}
x_1 + x_2 & \leq 4, \\
x_1 + 3x_2 & \leq 6,
\end{align*}
\]

\( x \geq 0, \quad \max f(x) = (c_1, c_2)^T x. \quad (8.2) \)

(a) (2 points) Write the problem in canonical form.

Solution. Introducing the slack variables, we have

\[
\begin{align*}
x_1 + x_2 + s_1 & = 4, \\
x_1 + 3x_2 + s_2 & = 6,
\end{align*}
\]

\( s \geq 0, \quad \max g(x, s) = (c_1, c_2, 0, 0)^T (x, s). \)

(b) (7 points) Rewrite \( f \) in three ways: \( f(s) \), \( f(x_1, s_2) \), and \( f(x_2, s_1) \).

Solution. To find \( f(s) \), we solve \( B \) for \( x \) in terms of \( s \):

\[
\begin{align*}
-x_1 - x_2 - s_1 & = -4 \\
x_1 + 3x_2 + s_2 & = 6
\end{align*}
\]

\( \implies \quad 2x_2 - s_1 + s_2 = 2, \quad (C.1) \)

\[
\begin{align*}
-3x_1 - 3x_2 - 3s_1 & = -12 \\
x_1 + 3x_2 + s_2 & = 6
\end{align*}
\]

\( \implies \quad -2x_1 - 3s_1 + s_2 = -6. \quad (C.2) \)

Then solving \( C \) for \( x \), we obtain

\[
\begin{align*}
x_1 & = \frac{-3s_1 + s_2 + 6}{2}, \\
x_2 & = \frac{s_1 - s_2 + 2}{2},
\end{align*}
\]

\[
f(s) = c_1 \left( \frac{-3s_1 + s_2 + 6}{2} \right) + c_2 \left( \frac{s_1 - s_2 + 2}{2} \right)
\]

\[
= \frac{s_1(-3c_1 + c_2)}{2} + \frac{s_2(c_1 - c_2)}{2} + 3c_1 + c_2. \quad (D.1)
\]
To find \( f(x_1, s_2) \), we solve (B.2) for \( x_2 \):

\[
x_2 = \frac{-x_1 - s_2 + 6}{3},
\]

\[
f(x_1, s_2) = c_1 x_1 + c_2 \left( \frac{-x_1 - s_2 + 6}{3} \right)
= x_1 \left( c_1 - \frac{c_2}{3} \right) + s_2 \left( -\frac{c_2}{3} \right) + 2c_2.
\]  
\( \text{(D.2)} \)

To find \( f(x_2, s_1) \), we solve (B.1) for \( x_1 \):

\[
x_1 = 4 - x_2 - s_1,
\]

\[
f(x_2, s_1) = c_1 (4 - x_2 - s_1) + c_2 x_2
= x_2 (-c_1 + c_2) + s_1 (-c_1) + 4c_1.
\]  
\( \text{(D.3)} \)

(c) (5 points) Consider the \((c_1, c_2)\) diagram above, which is divided by thick solid lines into four regions. Use your answer to (b) to show that in each of the four regions, the optimal solution has a different two of the four components of \((x, s)\) equal to zero, and identify which region corresponds to each pair.

**Solution.** From (8.2) we see that if \( c_1 < 0, c_2 < 0 \), \( f \) must be maximized for \( x = 0 \); otherwise it would be negative. (This is region A.) Similarly, by (D.1) we have that if

\[
-3c_1 + c_2 < 0 \quad \Rightarrow \quad c_2 < 3c_1
\]

\[
c_1 - c_2 < 0 \quad \Rightarrow \quad c_2 > c_1,
\]

then \( s = 0 \) for the function to be maximized. Therefore, we have that the lines are given by \( c_2 = c_1, c_2 = 3c_1 \), and that this is region C. By (D.2), we have that if

\[
c_1 - \frac{c_2}{3} < 0 \quad \Rightarrow \quad c_2 > 3c_1
\]

\[
-\frac{c_2}{3} < 0 \quad \Rightarrow \quad c_2 > 0, \quad \Rightarrow \quad c_2 > \max\{0, 3c_1\},
\]

then the maximum is given by \((x_1, s_2) = (0, 0)\). This is region D. Lastly, by (D.3) we have that if \(-c_1 + c_2 < 0, -c_1 < 0 \) \((c_1 > \max\{0, c_2\})\), then the maximum is given by \((x_2, s_1) = (0, 0)\). This is region B. The completed diagram is shown.

(d) (4 points) What happens when the \( c_j \) lie along the dark lines in the diagram?

**Solution.** First consider (8.2). When \( c_2 = 0 \), \( x_2 \) doesn’t contribute to \( f \). Hence in the case \( c_1 < 0, (x_1 = 0, x_2 = 0) \) is a maximum for any feasible \( x_2 \). So there are infinitely many optimal solutions. Similarly, for \( c_2 < 0, c_1 = 0, (x_1, x_2 = 0) \) is a maximum for any feasible \( x_1 \). Next consider (C.1). When \( c_1 = c_2, s_2 \) doesn’t contribute to \( f \). Hence in the case \( c_2 < 3c_1, (s_1 = 0, s_2 = 0) \) is a maximum for any feasible \( s_2 \). So there are infinitely many optimal solutions. Similarly, for \( c_2 = 3s_1, c_2 > c_1, (s_1, s_2 = 0) \) is a maximum for any feasible \( s_1 \). So on each of the lines, there are infinitely many optimal solutions.
Figure 8.2. Answer to \#5(c).