Homework Set 6 Solutions

1. Consider the following problem:

Maximize \( f(x) = -(x_1 - 2)^2 - (x_2 - 1)^2 \) on \( x \in [0, 2] \times [0, 2] \).

[In other words, the domain is the square with corners (0,0) and (2,2).]

(a) (2 points) Find the solution by inspection.

Solution. In the domain, \(- (x_1 - 2)^2 \leq 0\), with equality if and only if \( x_1 = 2 \), which is in the domain. Also, in the domain \(- (x_2 - 1)^2 \leq 0\), with equality if and only if \( x_2 = 1 \), which is also in the domain. So the maximum occurs at (2,1).

(b) (2 points) Find two conditions \( g_i(x) \leq b_i \) that (along with the requirement that \( x \geq 0 \)) define the square.

Solution. We must have that

\[
\begin{align*}
x_1 & \leq 2, \quad (A.1) \\
x_2 & \leq 2. \quad (A.2)
\end{align*}
\]

(c) (6 points) Verify that your solution satisfies the Kuhn-Tucker conditions. Interpret your solution in terms of inactive and active constraints.

Solution. Calculating (K1), we have

\[
\nabla(- (x_1 - 2)^2 - (x_2 - 1)^2) - y_1 \nabla(x_1) - y_2 \nabla(x_2) = \begin{pmatrix} -2(x_1 - 2) - y_1 \\ -2(x_2 - 1) - y_2 \end{pmatrix} \leq 0, \quad y \geq 0,
\]

\[
-2(x_1 - 2) - y_1 \leq 0, \quad (B.1)
\]

\[
-2(x_2 - 1) - y_2 \leq 0. \quad (B.2)
\]

Similarly, (K2) and (K3) become

\[
x_1[-2(x_1 - 2) - y_1] + x_2[-2(x_2 - 1) - y_2] = 0, \quad (C.1)
\]

\[
y_1(x_1 - 2) + y_2(x_2 - 2) = 0. \quad (C.2)
\]

With \( x_2 = 1 \), we see from (B.2) or (C.2) that \( y_2 = 0 \), so (C.1) becomes

\[
x_1[-2(x_1 - 2) - y_1] = 0,
\]

and since \( x_1 = 2, y_1 = 0 \), which also satisfies (B.1) and (C.2). Therefore, we see that (A.1) is active, as is (B.1). Also, (A.2) is inactive, but the corresponding condition (B.2) is active. In addition, both constraints that \( x \geq 0 \) are inactive.
2. Consider the following nonlinear programming problem:

\[
\max_x f(x) = x_1 + 2x_2 \quad \text{subject to} \quad \begin{align*}
2x_1^2 + x_2^2 & \leq 1, \\
-x_1 - 2x_2 & \leq -1, \\
x_1 & \geq 0.
\end{align*}
\]

(a) (3 points) Using a two-dimensional graph, illustrate the solution geometrically. In particular, indicate whether the solution occurs on the interior or on one of the boundaries (and if so, which one).

Solution. See Figure 1. Note that the solution occurs on the boundary \(2x_1^2 + x_2^2 = 1\).

![Figure 1. Solid curve: \(2x_1^2 + x_2^2 = 1\). Dashed line: \(x_1 - 2x_2 = -1\). Shaded region: feasible. Dotted lines: level sets of \(f(x)\) (increasing as thickness increases). Circle: optimal solution \(x\).](image)

Now we wish to solve the problem using our solution to (a) as a guide, but verifying everything algebraically.

(b) (9 points) By establishing contradictions, show that \(y_1 \neq 0\), \(x_2 \neq 0\), and \(x_1 \neq 0\) (in that order). Also verify that

\[
y_2 = \frac{x_2 - 4x_1}{4x_1 + x_2}. \tag{6.1}
\]
Solution. Constructing the components of the Kuhn-Tucker conditions, we have

\[ f(x) = x_1 + 2x_2, \]
\[ g_1(x) = 2x_1^2 + x_2^2 \leq 1, \quad \text{(D.1)} \]
\[ g_2(x) = x_1 - 2x_2 \leq -1, \quad \text{(D.2)} \]
\[ \nabla f - y_1 \nabla g_1 - y_2 \nabla g_2 = (1 - y_1(4x_1) - y_2, 2 - y_1(2x_2) + 2y_2) \leq 0. \quad \text{(K1)} \]

Now computing the slackness conditions, we have

\[ (1 - 4x_1y_1 - y_2)x_1 + (2 - 2x_2y_1 + 2y_2)x_2 = 0, \quad \text{(K2)} \]
\[ y_1(2x_1^2 + x_2^2 - 1) + y_2(x_1 - 2x_2 + 1) = 0. \quad \text{(K3)} \]

We first consider the case where \( y_1 = 0 \). Then the second component of (K1) becomes \( 2 + 2y_2 \leq 0 \), since \( y_2 \geq 0 \). So \( y_1 \neq 0 \), and by the slackness condition (D.1) must be satisfied as an equality:

\[ 2x_1^2 + x_2^2 = 1. \quad \text{(E)} \]

Next suppose that \( x_2 = 0 \). Then (D.2) cannot be satisfied. So \( x_2 \neq 0 \) and by the slackness conditions, the second component of (K1) is satisfied as an equality.

Now suppose that \( x_1 = 0 \), in which case \( x_2 = 1 \) from (E). In addition, the first component of (K1) becomes \( 1 - y_2 \leq 0 \), so \( y_2 \neq 0 \) and (D.2) is satisfied as an equality, which yields \( x_2 = 1/2 \), which is a contradiction. Hence \( x_1 \neq 0 \) and the first component of (K1) is also satisfied as an equality. Therefore, (K1) is replaced by

\[ 1 - 4x_1y_1 - y_2 = 0, \quad \text{(F.1)} \]
\[ 2 - 2x_2y_1 + 2y_2 = 0 \]
\[ 1 - x_2y_1 + y_2 = 0 \quad \text{(F.2)} \]

We multiply (F.1) by \( x_2 \) and (F.2) by \(-4x_1\) to solve for \( y_2 \):

\[ x_2 - 4x_1x_2y_1 - x_2y_2 = 0 \]
\[ -4x_1 + 4x_1x_2y_1 - 4x_1y_2 = 0 \]
\[ -4x_1 + 4x_1x_2y_1 - 4x_1y_2 = 0 \]
\[ y_2 = \frac{x_2 - 4x_1}{4x_1 + x_2}, \]

as required.

(c) (9 points) Show that assuming that \( y_2 \neq 0 \) leads to a contradiction. Verify that the maximum value of \( f \) is \( 3\sqrt{2}/2 \).

Solution. If \( y_2 \neq 0 \), (D.2) also holds as an equality, so we have

\[ x_1 = 2x_2 - 1 \implies x_2 = \frac{x_1 + 1}{2}, \quad \text{(G)} \]

which, when substituted into (6.1), yields

\[ y_2 = \frac{-4x_1 + (x_1 + 1)/2}{4x_1 + (x_1 + 1)/2} = \frac{1 - 7x_1}{1 + 9x_1}. \]
Therefore, since \(y_2 > 0\), we must have that \(1 - 7x_1 > 0\), so \(x_1 < 1/7\). Then combining (G) with (E), we have

\[
2x_1^2 + \left( \frac{x_1 + 1}{2} \right)^2 = 1
\]
\[
8x_1^2 + x_1^2 + 2x_1 + 1 = 4
\]
\[
9x_1^2 + 2x_1 - 3 = 0
\]
\[
x_1 = \frac{-2 \pm \sqrt{4 + 108}}{18} = \frac{-2 \pm 4\sqrt{7}}{18} = -\frac{1 + 2\sqrt{7}}{9},
\]
where we have chosen the root that makes \(x_1 > 0\). But \(2\sqrt{7} > 4\), so \(x_1 > 3/9\), which is not less than \(1/7\). So \(y_2 = 0\), in which case we have

\[
x_1 = \frac{1}{4y_1}, \quad x_2 = \frac{1}{y_1}
\]
from (F), which we may substitute into (E) to obtain

\[
2 \left( \frac{1}{4y_1} \right)^2 + \left( \frac{1}{y_1} \right)^2 = 1
\]
\[
\frac{1}{8y_1^2} + \frac{1}{y_1^2} = \frac{9}{8y_1^2} = 1
\]
\[
y_1 = \frac{2\sqrt{2}}{3}
\]
\[
x_1 = \frac{2\sqrt{2}}{12} = \frac{\sqrt{2}}{6},
\]
\[
x_2 = \frac{2\sqrt{2}}{3}.
\]

To double-check our result, we substitute these results into (D.2), obtaining

\[
\frac{\sqrt{2}}{6} - \frac{4\sqrt{2}}{3} = -\frac{7\sqrt{2}}{6} \leq -1,
\]
which checks. So the maximum is given by

\[
f \left( \frac{\sqrt{2}}{6}, \frac{2\sqrt{2}}{3} \right) = \frac{\sqrt{2}}{6} + 2 \left( \frac{2\sqrt{2}}{3} \right) = \frac{(1 + 8)\sqrt{2}}{6} = \frac{3\sqrt{2}}{2},
\]
as required.

3. (3 points) Let \(g(x)\) be convex. Prove that \(G(x, s) = g(x) + s\) is convex.

**Solution.** Let \((x, s) = (1 - \lambda)(x_1, s_1) + \lambda(x_2, s_2)\). We must show that

\[
G(x, s) \leq (1 - \lambda)G(x_1, s_1) + \lambda G(x_2, s_2)
\]
\[
g(x) + s \leq (1 - \lambda)[g(x_1) + s_1] + \lambda[g(x_2) + s_2]
\]
\[
g(x) \leq (1 - \lambda)g(x_1) + \lambda g(x_2),
\]
where we have used the definition of $s$. But since $g(x)$ is convex, the last equation above holds, so $G(x, s)$ is convex.

4. (6 points) Suppose that in the Kuhn-Tucker problem, the standard condition $x \geq 0$ is replaced by $0 \leq x \leq u$. Show that in this case, (K1) may be written as

$$\nabla f - \sum_{i=1}^{m} y_i \nabla g_i \leq z, \quad y \in \mathbb{R}^m, \quad z \in \mathbb{R}^n; \quad y \geq 0, \quad z \geq 0, \quad (K1')$$

and write similar conditions for (K2) and (K3).

**Solution.** Our new constraints are $x \leq u$. But we can simply add these constraints on to our existing constraints $g(x) \leq b$, as follows:

$$g'(x) = \left( \begin{array}{c} g(x) \\ x \end{array} \right) \leq \left( \begin{array}{c} b \\ u \end{array} \right) = b' \in \mathbb{R}^{m+n}.$$  

We now have $m + n$ Lagrange multipliers, which we write as follows:

$$y' = \left( \begin{array}{c} y \\ z \end{array} \right), \quad y \in \mathbb{R}^m, \quad z \in \mathbb{R}^n; \quad y \geq 0, \quad z \geq 0.$$  

Hence have that (K1) is replaced by

$$\nabla f - \sum_{i=1}^{m+n} y_i \nabla g_i' = \nabla f - \sum_{i=1}^{m} y_i \nabla g_i - \sum_{j=1}^{n} z_j \nabla x_j \leq 0.$$  

But $\nabla x_j = e_j$, so the second sum just becomes $z$. Hence we have

$$\nabla f - \sum_{i=1}^{m} y_i \nabla g_i - z \leq 0,$$

from which $(K1')$ immediately follows. Then $(K2)$ and $(K3)$ become

$$(\nabla f - \sum_{i=1}^{m} y_i \nabla g_i - z)^T x = 0, \quad (K2')$$

$$y'^T [g'(x) - b'] = y^T [g(x) - b] + z^T (x - u) = 0. \quad (K3')$$