Homework Set 4 Solutions

1. Since the optimal bundle $x^*$ depends on the price vector $p \geq 0$ and the budget $I$, we have that the optimal utility $U^* = U(x^*)$ can also be written as a function of $p$ and $I$: $U^*(p, I)$, where $U^*$ is called the indirect utility function.

(a) (5 points) Show that the indirect utility function is a decreasing function of all prices and an increasing function of income. As an intermediate step, you should show that

$$\frac{\partial x^*_j}{\partial p_j} = -\frac{x^*_j}{p_j}, \quad \frac{\partial x^*_j}{\partial p_i} = -\frac{x^*_i}{p_j}, \quad (4.1)$$

where the $x^*_j$ and $p_j$ are components of the optimal bundle and price vector, respectively.

Solution. We know from Homework Set 3, #5 that $x^*$ must lie on the budget line, so we have

$$p^T x^* - I = 0$$
$$\sum_{j=1}^n p_j x^*_j - I = 0$$

$$x^*_j = \frac{1}{p_j} \left( I - \sum_{i \neq j} p_i x^*_i \right). \quad (A)$$

Therefore, we have that

$$\frac{\partial x^*_j}{\partial p_j} = -\frac{1}{p_j^2} \left( I - \sum_{i \neq j} p_i x^*_i \right) = -\frac{x^*_j p_j}{p^2_j} = -\frac{x^*_j}{p_j}$$
$$\frac{\partial x^*_j}{\partial p_i} = \frac{x^*_i}{p_j},$$

which is exactly (4.1). Calculating our derivatives, we have

$$\frac{\partial U^*}{\partial p_i} = \nabla U^* \cdot \frac{\partial x^*}{\partial p_i}, \quad \frac{\partial U^*}{\partial I} = \nabla U^* \cdot \frac{\partial x^*}{\partial I}. \quad (B)$$

We know that $\nabla U^* > 0$, and since $x_j$ and $p_j$ are both positive, we have from (4.1) that $\frac{\partial x^*}{\partial p_i} < 0$, so their dot product is less than zero. Hence $U^*$ is a decreasing function of price. Similarly, from (A) we have that

$$\frac{\partial x^*_j}{\partial I} = \frac{1}{p_j} > 0 \quad (C)$$
for all $j$, so $\partial U^*/\partial I > 0$, and $U$ is an increasing function of income.

(b) (3 points) Show that

$$x_j^* = -\frac{\partial U^*/\partial p_j}{\partial U^*/\partial I}.$$  

**Solution.** Combining (4.1) and (C), we have

$$\frac{\partial x_j^*}{\partial p_i} = -\frac{x_i^*}{p_j} = -x_i^* \frac{\partial x_j^*}{\partial I} \implies \frac{\partial x^*}{\partial p_i} = -x_i^* \frac{\partial x^*}{\partial I}.$$  

Substituting this result into (B), we have

$$\frac{\partial U^*}{\partial p_i} = \nabla U^* \cdot \left(-x_i^* \frac{\partial x^*}{\partial I}\right) = -x_i^* \frac{\partial U^*}{\partial I}.$$  

Replacing $i$ by $j$ and rearranging, we have the desired result.

2. (6 points per part) For the following relations on the listed spaces, determine the conditions (if any) on the listed characteristic that:

- make the relation a preference relation,
- make the preference relation complete,
- make the complete preference relation strongly monotone.

In the case of a complete strongly monotone preference relation, write a corresponding utility function. (You may assume that any preference relation which satisfies all the other properties is continuous.)

(a) characteristic: set $X \subseteq \mathcal{R}^n$, $x \succeq y$ if and only if $\min_i x_i \geq \max_i y_i$.

**Solution.** For general $X$, this is not a preference relation, for it is not symmetric. In particular,

$$x \succeq x \implies \min_i x_i \geq \max_i x_i,$$

which occurs only if $\min_i x_i = \max_i x_i$, which implies that all the $x_i$ are the same. Hence this is only a preference relation if $X = \{x \in \mathcal{R}^n | x = t\mathbf{1}, t \in \mathcal{R}\}$. On this set, we may redefine the relation as

$$x \succeq y \text{ if and only if } t_x \geq t_y,$$

where $x = t_x \mathbf{1}$ and similarly for the other variables. Hence it is transitive, since

$$x \succeq y, \quad y \succeq z \implies t_x \geq t_y, \quad t_y \geq t_z \implies t_x \geq t_z \implies x \succeq z.$$  

Thus it is a preference relation, and it is complete because either $t_x \geq t_y$ or $t_y \geq t_x$ for all $x, y$, so either $x \succeq y$ or $y \succeq x$. It is also strongly monotone, since

$$x_j \succeq y_j, \quad x \neq y \implies t_x > t_y \implies x \succ y.$$
A corresponding utility function is \( U(x) = t_x \).

(b) characteristic: vector \( z \)

\( X = \{ x \in \mathbb{R}^n | x \geq 0 \}, \ x \succeq y \) if and only if the distance between \( x \) and \( z \) is greater than or equal to the distance between \( y \) and \( z \), for some fixed \( z \in \mathbb{R}^n \).

**Solution.** Though we could consider the actual distance, for simplicity we consider \( U(x) = |x - z|^2 \) instead, which we may do WLOG. The relation is a preference relation for any \( z \), since \( U(x) \geq U(y) \), so we have reflexivity. Moreover,

\[ x \succeq y, \ y \succeq z \implies U(x) \geq U(y), \ U(y) \geq U(z), \]

which implies that

\[ U(x) \geq U(z) \implies x \succeq z, \]

so the relation is transitive. It is complete for any \( z \) since we can compute (and hence compare) \( U(x) \) for any \( x \). For strong monotonicity, we must have

\[ x_j \geq y_j, \ x \neq y \implies x \succ y \implies U(x) > U(y). \]

If this is true for all \( x \) and \( y \), it must be true for \( x = y + \delta u \), where \( \delta > 0 \) and \( u \geq 0 \) is a unit vector. But

\[
U(y + \delta u) = \sum_{j=1}^{n} (y_j + \delta u_j - z_j)^2 = \sum_{j=1}^{n} (y_j - z_j)^2 + 2\delta u_j (y_j - z_j) + \delta^2 u_j^2 \\
= U(y) + \delta^2 + 2\delta \sum_{j=1}^{n} u_j (y_j - z_j).
\]

Taking the limit as \( \delta \to 0^+ \), we see that \( y_j - z_j \geq 0 \) for all \( j \) to guarantee that \( U(x) > U(y) \). Since \( y \in X \) is arbitrary, we see that \( y_j \) can be as small as zero, which means that \( z_j \leq 0 \) for all \( j \). So \( -z \in X \) for the preference relation to be strongly monotone, in which case \( U \) is the utility function.

(c) characteristic: set \( X \)

\[ X \subseteq \mathbb{R}^n, \ n \text{ odd}; \ x \succeq y \text{ if and only if } \prod_{i=1}^{n} x_i \geq \prod_{i=1}^{n} y_i. \]

**Solution.** Let

\[ U(x) = \prod_{i=1}^{n} x_i. \]

The relation is a preference relation for any \( X \), since \( U(x) \geq U(x) \), so we have reflexivity. Moreover,

\[ x \succeq y, \ y \succeq z \implies U(x) \geq U(y), \ U(y) \geq U(z), \]

which implies that

\[ U(x) \geq U(z) \implies x \succeq z, \]
so the relation is transitive. It is complete for any \( z \) since we can compute (and hence compare) \( U(x) \) for any \( x \). For strong monotonicity, we must have

\[
x_i \geq y_i, \quad x \neq y \implies x \succ y \implies U(x) > U(y).
\]

If this is true for all \( x \) and \( y \), it must be true for \( x = y + \delta u \), where \( \delta > 0 \) and \( u \geq 0 \) is a unit vector. But

\[
U(y + \delta u) = \prod_{i=1}^{n}(y_i + \delta u_j) = \prod_{i=1}^{n} y_i + \delta \sum_{j=1}^{n} u_j \prod_{i \neq j} y_i + O(\delta^2).
\]

Taking the limit as \( \delta \to 0^+ \), we see that the product of \( n - 1 \) terms must be non-negative for all \( j \) to guarantee that \( U(x) \geq U(y) \). Since \( n \) is odd, that means that either all the \( y_i \geq 0 \), or all the \( y_i \leq 0 \). So \( y \geq 0 \) or \( y \leq 0 \). Now suppose that \( y_i = 0 \), \( u = e_k \), \( k \neq i \). But then \( u_i = 0 \), so \( x_i = 0 \), and \( U(x) = U(y) \), \( x \neq y \). So to guarantee strong monotonicity, \( y_i \neq 0 \) for any \( i \). Hence \( X = \{ x \in \mathbb{R}^n | x > 0 \text{ or } x < 0 \} \).

3. Let \( f(x) \in C^\infty[a, b] \) (i.e., \( f(x) \) has infinitely many continuous derivatives in \( [a, b] \)). Moreover, let \( f(x) \) be strictly concave.

(a) (6 points) Show that wherever \( f''(x) \neq 0 \) for some \( x \in [a, b] \), \( f''(x) < 0 \).

**Solution.** The proof proceeds similarly to Homework Set 1, #2. If \( f \) is strictly concave, then for any \( x \in [a, b] \),

\[
(1 - \lambda)f(x_1) + \lambda f(x_2) < f((1 - \lambda)x_1 + \lambda x_2), \quad 0 < \lambda < 1.
\]

Letting \( h = x_2 - x_1 \), we have

\[
(1 - \lambda)f(x_1) + \lambda f(x_1 + h) < f((1 - \lambda)x_1 + \lambda(x_1 + h)) = f(x_1 + \lambda h).
\]

Since \( f(x) \in C^\infty[a, b] \), we may expand in a Taylor series about \( x_1 \):

\[
f(x_1 + h) = f(x_1) + hf'(x_1) + \frac{h^2}{2} f''(\xi), \quad \xi \in [x_1, x_1 + h] = [x_1, x_2],
\]

\[
f(x_1 + \lambda h) = f(x_1) + (\lambda h)f'(x_1) + \frac{(\lambda h)^2}{2} f''(\eta), \quad \eta \in [x_1, x_1 + \lambda h],
\]

where we have assumed that \( f''(x) \neq 0 \) everywhere in \( [x_1, x_2] \). Substituting the above into (E) and simplifying, we obtain

\[
(1 - \lambda)f(x_1) + \lambda \left[ f(x_1) + hf'(x_1) + \frac{h^2}{2} f''(\xi) \right] < f(x_1) + \lambda hf'(x_1) + \frac{(\lambda h)^2}{2} f''(\eta)
\]

\[
f(x_1) + h\lambda f'(x_1) + \frac{h^2}{2} \lambda f''(\xi) < f(x_1) + \lambda hf'(x_1) + \frac{(\lambda h)^2}{2} f''(\eta)
\]

\[
\lambda \frac{h^2}{2} f''(\xi) < \frac{(\lambda h)^2}{2} f''(\eta)
\]

\[
f''(\xi) < \lambda f''(\eta),
\]
where we have used the fact that \( \lambda \neq 0 \). But this must be true for \( x_2 \to x_1 \), which implies that \( \eta \) and \( \xi \) both tend to \( x_1 \). Hence we have

\[
(1 - \lambda)f''(x_1) < 0.
\]

Since \( 0 < \lambda < 1 \), we have that the coefficient of \( f'' \) is positive, so \( f''(x_1) < 0 \). But this must be true for \( x_2 \to x_1 \), which implies that \( \eta \) and \( \xi \) both tend to \( x_1 \). Hence we have

\[
(1 - \lambda)f''(x_1) < 0.
\]

(b) (3 points) Show that \( f''(x) = 0 \) only at isolated points in \( [a, b] \).

Solution. Suppose that \( f''(x) = 0 \) for \( x \in [x_1, x_2] \). Then \( f^{(n)}(x) = 0 \) for all \( x \in [x_1, x_2] \) as well, so

\[
f(x) = c_1 + c_2 x, \quad x \in [x_1, x_2],
\]

for some constants \( c_1 \) and \( c_2 \). But then

\[
(1 - \lambda)f(x_1) + \lambda f(x_2) = (1 - \lambda)(c_1 + c_2 x_1) + \lambda(c_1 + c_2 x_2) \\
= c_1 + c_2[(1 - \lambda)x_1 + \lambda x_2] \\
= f((1 - \lambda)x_1 + \lambda x_2),
\]

which contradicts the definition in (D).

4. (5 points) Prove Theorem CF1 in the other direction. In particular, let \( X \subseteq \mathbb{R}^n \), \( f : X \to \mathbb{R} \). Moreover, assume that for any \( x_i \in X \),

\[
\sum_{i=1}^{n} \lambda_i f(x_i) \leq f \left( \sum_{i=1}^{n} \lambda_i x_i \right), \quad \lambda_i \in [0, 1], \quad \sum_{i=1}^{n} \lambda_i = 1. \tag{4.2}
\]

Prove that \( \text{hyp } f \) is convex.

Solution. Suppose that we have two points \( (y_i, x_i) \in \text{hyp } f \), so \( y_i \leq f(x_i) \). To prove that \( \text{hyp } f \) is convex, we must prove that

\[
(y, x) = (1 - \lambda)(y_1, x_1) + \lambda(y_2, x_2) \in \text{hyp } f,
\]

which is equivalent to showing that \( y \leq f(x) \). Since \( y_j \leq f(x_j) \), we have

\[
y = (1 - \lambda)y_1 + \lambda y_2 \leq (1 - \lambda)f(x_1) + \lambda f(x_2).
\]

Without loss of generality, take \( \lambda_i = 0 \) for \( i > 2 \) in (4.2). Then we may rewrite (4.2) as

\[
(1 - \lambda)f(x_1) + \lambda f(x_2) \leq f(x), \quad x \equiv (1 - \lambda)x_1 + \lambda x_2.
\]

Putting the above two expressions together, we have \( y \leq f(x) \), as required. So the convex combination of any two points in \( \text{hyp } f \) is in \( \text{hyp } f \), so \( \text{hyp } f \) is convex.