Homework Set 2 Solutions

1. Let \( X_i \) be a bounded, connected set in \( \mathbb{R}^2 \) for \( i = 1, 2, \ldots, m \). Moreover, let all the \( X_i \) be disjoint:

\[
X_i \cap X_j = \emptyset, \quad i \neq j,
\]

and let \( X_{\cup} \) be the union of all the \( X_i \):

\[
X_{\cup} = \bigcup_{i=1}^{m} X_i.
\]

First, we let \( m = 2 \).

(a) (10 points) Show that every \( x \in \text{conv}(X_{\cup}) \) can be written as the convex combination of two boundary points of \( X_{\cup} \).

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure21.png}
\caption{\( X_j \) are white, and the convex hull is shaded. Left: \( x \in X_1 \). Right: \( x \notin X_i \).}
\end{figure}

Solution. Since \( x \in \text{conv}(X_{\cup}) \), let

\[
x = z(\lambda_+), \quad z(\lambda) = (1 - \lambda)x_1 + \lambda x_2
\]

for some \( \lambda \in [0, 1] \). (Refer to Figure 2.1.) In the proof below, we assume without loss of generality that \( x_i \in X_i \), though the same argument if both \( x_i \) are in the same set. Now
consider additional points on the line with $\lambda > 1$. Since $X_2$ is bounded, there must exist some $\lambda_+ \geq 1$ such that $z(\lambda) \notin X_2$, $\lambda > \lambda_+$. (The fact that $X_2$ is closed guarantees that it contains its boundary point, so $z(\lambda_+) \in X_2$.) Similarly, there must exist some $\lambda_- \leq 0$ such that $z(\lambda) \notin X_1$, $\lambda < \lambda_-$. But then we have the following two boundary points:

$$z(\lambda_+) = (1 - \lambda_+)x_1 + \lambda_+x_2, \quad z(\lambda_-) = (1 - \lambda_-)x_1 + \lambda_-x_2,$$

from which we can create the following combination:

$$\frac{\lambda_+}{\lambda_+ + |\lambda_-|} z(\lambda_-) - \frac{|\lambda_-|}{\lambda_+ + |\lambda_-|} z(\lambda_+) = \lambda_+(1 - \lambda_-)x - \lambda_-(1 - \lambda_+)x = (\lambda_+ - \lambda_-)x$$

where we have highlighted the fact that $\lambda_- \leq 0$. However, we note that both coefficients sum to 1 and are bounded between 0 and 1, so defining $\alpha$ to be the coefficient of $z(\lambda_+)$, we have

$$x = (1 - \alpha)z(\lambda_-) + \alpha z(\lambda_+)$$

for some $\alpha \in [0, 1]$, as required.

(b) (4 points) Provide a counterexample to show that the result in (a) does not hold if one of the $X_i$ is not connected.

![Figure 2.2](image)

**Figure 2.2.** The shaded area can be expressed as a convex combination of boundary points.

*Solution.* Refer to Figure 2.2, where $X_2 = X_2^a \cup X_2^b$ is not connected. Note that points in the shaded regions can be written as convex combinations of boundary points, because they form the convex hull of the union of any pair of the triangles. But the region marked (*) cannot, since those points do not lie on the line between boundary points of the triangles.
(c) (2 points) Use your answer to (b) to show that the result in (a) doesn’t hold if \( m > 2 \).

Solution. Let \( X_2 = X_2^a, X_3 = X_2^b \) in the example above. Then we have three connected sets, and the counterexample still holds.

2. (6 points) Let \( S \) be any closed, nonconvex polygon. Show that \( \text{conv} \ S \) has a greater area than \( S \), but a smaller perimeter.

Solution. Let \( S \) have \( n \) vertices. Label the vertices of the polygon \( x_1, x_2, \ldots \), where consecutive indices are connected by an edge (see Figure 5). There must be some pair of vertices \( x_i, x_j \) such that

\[(1 - \lambda)x_i + \lambda x_j \notin S, \text{ for some } \lambda \in [0, 1]\]  

or else the polygon would be convex. Moreover, there must be some \( i \) where \( j = (i + 2) \mod n \). (This is not required for the analysis, but useful for our algorithm.) These two points are two of the vertices of a triangle \( T \); the third is the vertex \( x_k \) between them.

![Figure 5](image)

Now we know that \( S \subset S' \subseteq \text{conv} \ S \), where \( S' \) is \( S \cup T \). But the area of \( S' \) is greater than the area of \( S \), since it also includes the area of \( T \). And the perimeter of \( S' \) is less than the perimeter of \( S \), since by the Triangle Inequality,

\[|x_i - x_j| < |x_i - x_k| + |x_k - x_j|,\]

where the inequality is strict, since otherwise (B) would be violated. Now continue the process of \( S' \), adding edges until the process terminates, at which point \( S' = \text{conv} \ S \) since all convex combinations are included in it.

3.
(a) (6 points) Prove that the closure of a convex set $S$ is convex.

**Solution.** Let $x_1, y_1 \in \text{cl} S$. Since $x_1, y_1 \in \text{cl} S$, they are limit points, so there must exist some points $x_2 \in B_\delta(x_1), y_2 \in B_\delta(y_1)$ that are also in $S$. We must show that $z_1 = (1 - \lambda)x_1 + \lambda y_1 \in \text{cl} S$, where $\lambda \in [0, 1]$. But this means we must show that there exists some $z_2 \in B_\delta(z_1)$ that is also in $S$. Since $S$ is convex, $z_2 = (1 - \lambda)x_2 + \lambda y_2 \in S$. But

$$|z_1 - z_2| = |(1 - \lambda)x_1 + \lambda y_1 - [(1 - \lambda)x_2 + \lambda y_2]| = |(1 - \lambda)(x_1 - x_2) + \lambda(y_1 - y_2)|$$

$$\leq |(1 - \lambda)(x_1 - x_2)| + |\lambda(y_1 - y_2)| = (1 - \lambda)|x_1 - x_2| + \lambda|y_1 - y_2|,$$

where we have used the Triangle Inequality and the fact that $\lambda$ and $1 - \lambda$ are both non-negative. But then using the definitions of $x_2$ and $y_2$, we have

$$|z_1 - z_2| \leq (1 - \lambda)\delta + \lambda\delta = \delta,$$

as required.

(b) (4 points) Provide a counterexample to show that the converse is not true.

**Solution.** The converse is: if $\text{cl} S$ is convex, then $S$ is convex. Let $S$ be the disk of radius 2, except for the origin:

$$S = \{(x, y)|0 < x^2 + y^2 \leq 4\}.$$

Then $\text{cl} S$ is the entire disk, which by notes in class is convex. But $S$ is not convex, since any line segment containing the origin will contain a point not in the set. For instance, if $x_1 = (0, 1), x_2 = (0, -1)$, then

$$(1 - \lambda)x_1 + \lambda x_2 \notin S \text{ for } \lambda = \frac{1}{2}.$$

4. Let $X$ be a set of $N$ points in $\mathbb{R}^n$, $N > n$. From Caratheodory’s Theorem we have that every $z \in \text{conv} X$ can be written as a convex combination of $m$ vectors in $X$, $m \leq n + 1$. Call any such set of $m$ vectors $X(z)$.

(a) (2 points) Find conditions (if any) such that the same set $W = X(z)$ in Caratheodory’s Theorem for every $z \in \text{conv} X$.

**Solution.** By the definition of $X(z)$, $z \in \text{conv} X(z)$. If there exists a set $W$ that equals $X(z)$ for all $z \in \text{conv} X$, then $\text{conv} X \subseteq \text{conv} W$. But since $W \subseteq X$, $\text{conv} W \subseteq \text{conv} X$. Hence $\text{conv} W = \text{conv} X$.

(b) (6 points) Find conditions (if any) such that the convex-combination representation of $z$ in Caratheodory’s Theorem is unique for every $z \in \text{conv} X$.

**Solution.** First consider the case $N > n + 1$, in which case there are at least two sets $X(z)$ of size $n+1$, which we call $X_1$ and $X_2$. If each $z \in \text{conv} X$ has a unique representation, then $\text{conv} X_1$ and $\text{conv} X_2$ must be disjoint. (Otherwise, any $z \in (\text{conv} X_1 \cap \text{conv} X_2)$ would have a convex-combination representation where $X_1 = X(z)$ and $X_2 = X(z)$.) Proceeding
similarly through all such sets $X_j$, we have that $\text{conv} X$ must be a union of disjoint sets. But by definition, $\text{conv} X$ must be connected, which is a contradiction.

Considering the case $N = n + 1$, we see that there is only one such set $X(z)$ (namely $X$), and hence the representation in Caratheodory’s Theorem is unique.