Updates

1. Office hours WILL be held on Tuesday, Nov. 3.

Homework Set 8 (Second Revision)

Read section 2.1, the intro to chapter 5, and section A-5, 5.1, 5.2.

Extreme Points

1. (4 points) In class, we listed the following two definitions of an extreme point \( z \) of a convex set \( X \):

   (a) \( z \) is not a convex combination of two other points in \( X \).
   (b) there do not exist two distinct points \( x, y \in X \) such that \( z = (1 - \lambda)x + \lambda y \), \( \lambda \in (0, 1) \).

   Prove that these two definitions are equivalent.

2. (2 points per part) Let \( S_1 \) and \( S_2 \) be disjoint, bounded, closed, convex regions in \( \mathbb{R}^2 \), \( N_j \) be the number of extreme points of \( S_j \), \( N_1 \geq N_2 \), \( N_2 \) finite. Let \( S' \) be the convex hull of \( S_1 \) and \( S_2 \), \( N' \) be the number of extreme points of \( S' \). Draw figures illustrating these cases. You do not need to go into a lengthy proof as long as your figures are clear.

   (a) \( N_1 < N' < \infty \)
   (b) \( N_1 < N' = \infty \)
   (c) \( N' < N_1 < \infty \)
   (d) \( N' < N_1 = \infty \)

Linear Programming Theory

3. Consider the maximum problem in canonical form:

   \[
   \max c^T x, \quad A x = b, \quad A \in \mathbb{R}^{m \times n}, \quad c \in \mathbb{R}^n, \quad x \geq 0 \in \mathbb{R}^n, \quad b \in \mathbb{R}^m. \tag{8.1}
   \]

   (a) (2 points) Prove that any convex combination of feasible basic solutions to (8.1) is also feasible.
   (b) (2 points) Prove that any convex combination of optimal basic solutions to (8.1) is also optimal.
4. Consider the following problem:
\[
\begin{pmatrix} 2 & 0 & 3 \\ 1 & 1 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 12 \\ 5 \end{pmatrix}, \quad \mathbf{x} \geq \mathbf{0}.
\]
(a) (4 points) Find all feasible basic solutions to the problem.
(b) (2 points) Find the optimal basic solution if we wish to maximize
\[U(\mathbf{x}) = 3x_1 - 2x_2 + x_3.\]

5. Consider the standard maximum problem
\[
x_1 + x_2 \leq 4, \quad \mathbf{x} \geq \mathbf{0}, \quad \max_{\mathbf{x}} f(\mathbf{x}), \quad f(\mathbf{x}) = (c_1, c_2)^T \mathbf{x}. \tag{8.2}
\]
(a) (2 points) Write the problem in canonical form.
(b) (7 points) Rewrite \(f\) in three ways: \(f(\mathbf{s})\), \(f(x_1, s_2)\), and \(f(x_2, s_1)\).
(c) (5 points) Consider the \((c_1, c_2)\) diagram above, which is divided by thick solid lines into four regions. Use your answer to (b) to show that in each of the four regions, the optimal solution has a different two of the four components of \((\mathbf{x}, \mathbf{s})\) equal to zero, and identify which region corresponds to each pair.
(d) (4 points) What happens when the \(c_j\) lie along the dark lines in the diagram?