Homework Set 1

Read chapter 2, the intro to chapter 5, and section A-5.

**WARNING**: A picture is not a proof. A picture *illustrates* a proof.

**Convex Sets**

1. For each of the following sets $S \in \mathbb{R}^2$, do each of the following, illustrating subcases as necessary:
   - sketch $S$ in distinguished cases corresponding to different regions of $\alpha$,
   - algebraically determine the conditions on $\alpha$ (if any) under which $S$ is closed,
   - algebraically determine the conditions on $\alpha$ (if any) under which $S$ is convex, and
   - in any case where $S$ is not convex, draw a line segment of convex combinations not contained in $S$.

   The proof of closure does not need to be as rigorous as the proof of convexity.

   (a) (5 points) $S = \{ x \geq 0, y \geq 0, xy > \alpha, \alpha \in \mathbb{R} \}$
   (b) (9 points) $S = \{ y \geq 0, x^2 + y^2 < 1 \}$
   (c) (5 points) $S = \{ y \geq 1 + \alpha |x|, \alpha \in \mathbb{R} \}$

2. (13 points) Consider the following region $X$:

   $$ X = \{(x,y) | x \geq 0, 0 \leq y \leq f(x)\}, \quad f(x) \in C^2[0, b], \quad f(0) > 0, $$

   where $b$ is the smallest positive zero of $f$. (Here $C^2[0, b]$ is the set of functions with continuous second derivatives on $[0, b]$.) Determine conditions on $f(x)$ such that $X$ is convex.

3. (4 points) Let $H = \{ x \in \mathbb{R}^n | u^T x = c \}$ for some given $u \in \mathbb{R}^n$, $c \in \mathbb{R}$. (Such a set is called a *hyperplane* in $\mathbb{R}^n$.) Show that $H$ has no interior points.

4. (4 points) Either prove the following statement, or provide a general example which disproves it. Again, just a diagram is not enough.

   Let $S_1$ and $S_2$ be convex sets that intersect at infinitely many points with $\text{int } S_1 \neq \phi$, $\text{int } S_2 \neq \phi$. Then $\text{int} (S_1 \cap S_2) \neq \phi$. 