The Kuhn-Tucker Conditions (Revised)

Suppose we wish to solve the following problem:

$$\max f(x) \text{ subject to } g(x) \leq b, \quad x \geq 0,$$

where $x \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, $g : \mathbb{R}^n \to \mathbb{R}^m$.

If we let $y \in \mathbb{R}^m$, $y \geq 0$ be the Lagrange multipliers and $s \in \mathbb{R}^m$, $s \geq 0$ be the slack variables, we may construct the function

$$F(x, s) = f(x) - \sum_{i=1}^m y_i [g_i(x) + s_i - b_i].$$

Optimizing this problem yields the following equations:

$$\nabla F(x, s) \leq 0, \quad (L1)$$

$$x, s \geq 0, \quad (L2)$$

$$[\nabla F(x, s)]^T (x, s) = 0, \quad (L3)$$

where $\nabla$ includes derivatives with respect to $x_j$ and $s_i$.

Rewriting (L1)–(L3) in the original variables yields the Kuhn-Tucker conditions, which say that the maximum is given by the point $x$ that satisfies

$$\nabla f(x) - \sum_{i=1}^m y_i \nabla g_i(x) \leq 0, \quad (K1)$$

$$\left[\nabla f(x) - \sum_{i=1}^m y_i \nabla g_i(x)\right]^T x = 0, \quad (K2)$$

$$y^T [g(x) - b] = 0, \quad (K3)$$

as well as the constraints on $x$ in (1).

Equations (K2) and (K3) are called complementary slackness conditions. Recall that due to the signs of the terms, each summand in (K2) and (K3) must be zero. For example,

$$y_i [g_i(x) - b_i] = 0 \text{ for every } i.$$