Homework Set 6 Solutions

1. For each pair of matrices \( \{ A, B \} \), find an elementary matrix \( E \) such that \( EA = B \). Also, explain in words what row operations each elementary matrix performs.

\[
A_1 = \begin{pmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 8 & 4 \\ -2 & 7 & 2 \end{pmatrix}, \quad (a)
\]

\[
A_2 = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 2 & 6 \\ 2 & 4 \end{pmatrix}, \quad (b)
\]

\[
A_3 = \begin{pmatrix} 1 & 7 \\ 0 & 3 \end{pmatrix}, \quad B_3 = \begin{pmatrix} 0 & 3 \\ 1 & 7 \end{pmatrix}, \quad (c)
\]

 Solution. To obtain \( B_1 \) from \( A_1 \), we must add the second row to twice the third row to obtain the new second row. The first and third rows remain unchanged, so we obtain

\[
E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}.
\]

To obtain \( B_2 \) from \( A_2 \), we multiply the first row by 2. The second row remains unchanged, so we obtain

\[
E_2 = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}.
\]

To obtain \( B_3 \) from \( A_3 \), we interchange the rows, so we obtain

\[
E_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\]

2. Let \( A \in \mathbb{R}^{3 \times 3} \) and suppose that

\[
\mathbf{a}_1 = 3\mathbf{a}_2 - 2\mathbf{a}_3. \quad (6.1)
\]

Will the system \( Ax = \mathbf{0} \) have a nontrivial solution? Is \( A \) nonsingular? Explain your answer.

 Solution. By the definition of matrix multiplication,

\[
Ax = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3.
\]
Now let \( \mathbf{x} = (1, -3, 2)^T \). Then \( A\mathbf{x} = a_1 - 3a_2 + 2a_3 \). But by (6.1), this is 0. Thus \( A\mathbf{x} = 0 \) has a nontrivial solution and \( A \) is singular. (Another possibility is \( \mathbf{x} = (-1, 3, -2)^T \), since in that case \( A\mathbf{x} = 0 \) by (6.1) as well.)

3. Let \( A \) and \( B \) be \( n \times n \) matrices and let \( C = AB \). Show that if \( B \) is singular, then \( C \) must be singular.

**Solution.** If \( B \) is singular, then \( B\mathbf{x} = 0 \) for some nonzero \( \mathbf{x} \). But then \( C\mathbf{x} = (AB)\mathbf{x} = A(B\mathbf{x}) = A\mathbf{0} = 0 \) for the same nonzero \( \mathbf{x} \), and hence \( C \) is singular.

4. Let

\[
C = \begin{pmatrix}
-1 & 1 & 1 \\
2 & 3 & 3 \\
3 & 4 & 5
\end{pmatrix}
\]

(a) Calculate \( C^{-1} \).

**Solution.** Using the augmented matrix notation, we have

\[
\begin{align*}
a \begin{pmatrix}
-1 & 1 & 1 & 1 & 0 & 0 \\
2 & 3 & 3 & 0 & 1 & 0 \\
3 & 4 & 5 & 0 & 0 & 1
\end{pmatrix} & \sim d = -3a + b \begin{pmatrix}
5 & 0 & 0 & -3 & 1 & 0 \\
0 & 5 & 5 & 2 & 1 & 0 \\
0 & 7 & 8 & 3 & 0 & 1
\end{pmatrix} \\
b & \sim e = 2a + b \begin{pmatrix}
5 & 0 & 0 & -3 & 1 & 0 \\
0 & 5 & 5 & 2 & 1 & 0 \\
0 & 7 & 8 & 3 & 0 & 1
\end{pmatrix}
\end{align*}
\]

Then dividing the last two rows by 5, we have that

\[
C^{-1} = \frac{1}{5} \begin{pmatrix}
-3 & 1 & 0 \\
1 & 8 & -5 \\
1 & -7 & 5
\end{pmatrix} \quad (A)
\]

(b) Use your answer to (a) to show that the solution of \( C\mathbf{x} = \mathbf{b} \) for the following cases is the vector listed:

\[
\begin{pmatrix}
5 \\
5 \\
5
\end{pmatrix} \Rightarrow \mathbf{x} = \begin{pmatrix}
-2 \\
4 \\
-1
\end{pmatrix}; \quad \begin{pmatrix}
10 \\
0 \\
0
\end{pmatrix} \Rightarrow \mathbf{x} = \begin{pmatrix}
-6 \\
2 \\
2
\end{pmatrix}
\]

**Solution.** In each case we have that \( \mathbf{x} = C^{-1}\mathbf{b} \), so using (A), we have in each case

\[
\begin{align*}
\mathbf{x} &= \frac{1}{5} \begin{pmatrix}
-3 & 1 & 0 \\
1 & 8 & -5 \\
1 & -7 & 5
\end{pmatrix} \begin{pmatrix}
5 \\
5 \\
5
\end{pmatrix} = \begin{pmatrix}
-2 \\
4 \\
-1
\end{pmatrix}, \\
\mathbf{x} &= \frac{1}{5} \begin{pmatrix}
-3 & 1 & 0 \\
1 & 8 & -5 \\
1 & -7 & 5
\end{pmatrix} \begin{pmatrix}
10 \\
0 \\
0
\end{pmatrix} = \begin{pmatrix}
-6 \\
2 \\
2
\end{pmatrix}.
\end{align*}
\]
5. Consider the matrix

\[ M = \begin{pmatrix} -3 - \lambda & -4 \\ -2 & 5 - \lambda \end{pmatrix}. \]

(a) Calculate \( \det M \).

**Solution.**

\[
\begin{vmatrix} -3 - \lambda & -4 \\ -2 & 5 - \lambda \end{vmatrix} = (-3 - \lambda)(5 - \lambda) - (-4)(-2) = -23 - 2\lambda + \lambda^2.
\]

(b) Find the value(s) of \( \lambda \) such that the \( \det M = 0 \).

**Solution.**

\[
\lambda^2 - 2\lambda - 23 = 0 \quad \Rightarrow \quad \lambda = \frac{-(-2) \pm \sqrt{4 - 4(1)(-23)}}{2(1)} = 1 \pm \sqrt{24} = 1 \pm 2\sqrt{6}.
\]

6. Prove that if a row or a column of an \( n \times n \) matrix consists entirely of zeroes, then \( \det A = 0 \).

**Solution.** Suppose that the \( i \)th row is all zeroes, so \( a_{ij} = 0 \) for every \( j \). But then given the fact that we can expand along that row, we have

\[
\det A = \sum_{j=1}^{n} a_{ij} \det A_{ij} = 0.
\]

Alternatively, suppose that the \( j \)th column is all zeroes, so \( a_{ij} = 0 \) for every \( i \). But then given the fact that we can expand along that column, we have

\[
\det A = \sum_{i=1}^{n} a_{ij} \det A_{ij} = 0.
\]

7. Calculate \( \det B \), where

\[
B = \begin{pmatrix} 2 & 1 & 5 & 0 \\ 4 & -3 & 1 & 1 \\ 0 & -1 & 2 & 0 \\ 3 & 2 & -1 & 0 \end{pmatrix}.
\]

**Solution.** Expanding by the fourth column, we have

\[
\det B = \begin{vmatrix} 2 & 1 & 5 \\ 4 & -3 & 1 \\ 0 & -1 & 2 \\ 3 & 2 & -1 \end{vmatrix} = 2 \begin{vmatrix} 1 & 5 \\ -1 & 2 \end{vmatrix} + 3 \begin{vmatrix} 1 & 5 \\ -1 & 2 \end{vmatrix} = 2(1 - 4) + 3(2 + 5) = 15.
\]
where in the second step we expanded by the first column.

8. Let \( A \in \mathcal{R}^{(2n+1) \times (2n+1)} \). Show that \( A^2 = -I \) has no solution.

\textit{Solution.} Taking the determinant of both sides, we have

\[
\det(A^2) = \det(-I)
\]

\[
(\det A)(\det A) = (-1)^{2n+1} \det I,
\]

\[
(\det A)^2 = -1,
\]

where we have used the fact that \( n \) is odd. But \( \det A \) must be real, so this equation has no solution.

9. Suppose that \( A, B, \) and \( C \) are \( n \times n \) matrices such that \( \det A = -1/2, \det B = 0, \) and \( \det C = 2/3 \). Calculate the following determinants:

(a) \( \det C^{-1} \)

\textit{Solution.} \( \det C^{-1} = (\det C)^{-1} = 3/2. \)

(b) \( \det(B^T) \)

\textit{Solution.} \( \det(B^T) = \det B = 0. \)

(c) \( \det(C^T) \)

\textit{Solution.} \( \det(C^T) = (\det C)(\det A)(\det B) = (2/3)(0)(-1/2) = 0. \)

(d) \( \det(2A^{-T}C) \)

\textit{Solution.}

\[
\det(2A^{-T}C) = 2^n(\det A^{-1})(\det C) = \frac{2^n(2/3)}{\det A} = -\frac{2^{n+2}}{3}.
\]

10. Let

\[
A = \begin{pmatrix} 4 & -1 \\ 2 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix}.
\]

Show by direct calculation that \( \det AB = (\det A)(\det B). \)

\textit{Solution.}

\[
\det A = \begin{vmatrix} 4 & -1 \\ 2 & 1 \end{vmatrix} = 4 - (-2) = 6, \quad B = \begin{vmatrix} 1 & 2 \\ 3 & 5 \end{vmatrix} = 5 - 6 = -1,
\]

\[
\det AB = \begin{vmatrix} 1 & 3 \\ 5 & 9 \end{vmatrix} = 9 - 15 = -6 = (\det A)(\det B).
\]