1. Consider the *series* RLC circuit shown in the figure. There is an initial voltage on the capacitor of $V_0$.

(a) Use the fact that the sum of the voltages around the loop must be zero to obtain the ODE

$$L\ddot{I} + R\dot{I} + \frac{I}{C} = 0.$$  \hspace{1cm} (4.1a)

*Solution.* From Homework Set 1, we know that the voltage through the inductor, resistor, and capacitor are described by $V_L = LI$, $V_R = IR$, $\dot{V}_C = I/C$, respectively. Integrating the last equation, we have that

$$V_C = \int_0^t \frac{I}{C} \, dt + V_0,$$

where we have used the fact that the initial voltage on the capacitor is $V_0$. Since the voltage around the loop must be zero, we have

$$L\dot{I} + IR + \int_0^t \frac{I}{C} \, dt + V_0 = 0.$$ \hspace{1cm} (A)

Taking the derivative of (A) with respect to $t$ yields (4.1a).

Initially, the current is $I_0$.

(b) Show that the resulting initial conditions are

$$I(0) = I_0, \hspace{1cm} \dot{I}(0) = -\frac{V_0 + RI_0}{L}. \hspace{1cm} (4.1b)$$
Solution. The first equation follows trivially from the problem statement. Substituting this result and the initial condition for the voltage into (A) evaluated at $t = 0$, we have

$$L \dot{I}(0) + I(0)R + V_0 = 0$$
$$\dot{I}(0) = - \frac{V_0 + RI_0}{L}.$$

(c) Under what conditions will the circuit be underdamped? overdamped? critically damped?

Solution. Substituting $I = e^{\lambda t}$ into (4.1a), we obtain

$$L\lambda^2 + R\lambda + \frac{1}{C} = 0 \implies \lambda_{\pm} = -\frac{R}{2L} \pm \frac{1}{2L} \sqrt{R^2 - \frac{4L}{C}}.$$

Therefore, our solutions are of the form $I = c_+ e^{\lambda_+ t} + c_- e^{\lambda_- t}$. The solutions are overdamped if the square root is real, so when $R^2 > 4L/C$. The solutions are underdamped if the square root is imaginary, so when $R^2 < 4L/C$. The solutions are critically damped when $R^2 = 4L/C$.

(d) Solve the system when $R = 2$, $C = 1/5$, $L = V_0 = I_0 = 1$.

Solution. Substituting our parameters into (B), we obtain

$$\lambda_{\pm} = -1 \pm \frac{1}{2} \sqrt{4 - 20} = -1 \pm 2i \implies I = e^{-t}(c_1 \cos 2t + c_2 \sin 2t).$$

Solving the initial conditions using our parameters, we obtain

$$I(0) = c_1 = I_0 = 1$$
$$\dot{I}(0) = -c_1 + 2c_2 = -\frac{V_0 + RI_0}{L} = -(1 + 2) = -3$$
$$2c_2 = -2$$
$$c_2 = -1$$
$$I(t) = e^{-t}(\cos 2t - \sin 2t),$$

where in the second line we have used the trick for calculating the derivative given in class.

2. A mass weighing 32 pounds stretches a spring 2 feet.

(a) Determine the amplitude and period of motion if the mass is initially released from a point 1 foot above the equilibrium position with an upward velocity of 2 ft/s.

Solution. By notes in class, we have that the spring constant is given by

$$k = \frac{\text{weight}}{\text{stretch}} = \frac{32 \text{ lb}}{2 \text{ ft}} = 16 \frac{\text{lb}}{\text{ft}} = 16 \frac{\text{slug}}{\text{s}^2}. $$
Moreover, we have that the mass is given by
\[ M = \frac{\text{weight}}{g} = \frac{32 \text{ lb}}{32 \text{ ft/s}^2} = 1 \text{ slug}. \]

Then the equation to solve becomes
\[ \ddot{x} + 16x = 0. \] (D.1)

Moreover, since “up” corresponds to compression, or negative \( x \), we have
\[ x(0) = -1, \quad \dot{x}(0) = -2, \] (D.2)

where feet is the underlying unit of distance. Substituting \( x = e^{\lambda t} \) into (D), we have
\[ \lambda^2 + 16 = 0 \quad \Rightarrow \quad \lambda_{\pm} = \pm 4i \]
\[ x(t) = c_1 \cos 4t + c_2 \sin 4t \]
\[ x(0) = c_1 = -1 \]
\[ \dot{x}(0) = 4c_2 = -2 \quad \Rightarrow \quad c_2 = -\frac{1}{2} \]
\[ x(t) = -\cos 4t - \frac{\sin 4t}{2} = \sqrt{1 + \frac{1}{4}} \cos(4t - \phi), \quad \tan \phi = \frac{1}{2}. \]

Then keeping in mind that \( c_1 < 0 \), we have
\[ x(t) = \frac{\sqrt{5}}{2} \cos(4t - \phi), \quad \phi = \pi + \tan^{-1} \frac{1}{2} \approx 3.61 \]

Hence the amplitude is \( \sqrt{5}/2 \) feet. In one period \( T \), \( 4T = 2\pi \), so \( T = \pi/2 \).

(b) How many complete cycles will the mass have made at the end of \( 4\pi \) seconds?

*Solution.* The number of cycles is given by
\[ \frac{4\pi}{T} = \frac{4\pi}{\pi/2} = 8. \]

3. Solve \( y'' - 16y = 2e^{4x} \)

by the method of undetermined coefficients.

*Solution.* Substituting \( y = e^{\lambda x} \) to find the homogeneous solution, we have
\[ \lambda^2 - 16 = 0 \quad \Rightarrow \quad \lambda = \pm 4 \quad \Rightarrow \quad x = c_1 e^{4x} + c_2 e^{-4x}. \]

Therefore, we see that the right-hand side is a multiple of a homogeneous solution, so we need to try a particular solution of the form
\[ y_p(x) = A xe^{4x}. \]
Substituting in this form, we have $y_p' = A(4x + 1)e^{4x}$, so we obtain
\[
\frac{d(A(4x + 1)e^{4x})}{dx} - 16xe^{4x} = 2e^{4x}
\]
\[
A[4(4x + 1) + 4]e^{4x} - 16Axe^{4x} = 2e^{4x}
\]
\[
8A = 2 \implies A = \frac{1}{4}
\]
\[
y(x) = c_1e^{4x} + c_2e^{-4x} + \frac{xe^{4x}}{4}.
\]

4. Find the solution to
\[
y^{(3)} + 5\dot{y} + 6\ddot{y} = 4e^{-t}, \quad y(0) = 0, \quad \dot{y}(0) = 1, \quad \ddot{y}(0) = 3.
\]

**Solution.** By substituting $y = e^{\lambda t}$, we can obtain the homogeneous solution, where $\lambda$ solves
\[
\lambda^3 + 5\lambda^2 + 6\lambda = \lambda(\lambda^2 + 5\lambda + 6) = \lambda(\lambda + 2)(\lambda + 3) = 0.
\]
Thus, we have
\[
y_h(t) = c_1 + c_2e^{-2t} + c_3e^{-3t}.
\]
The right-hand side doesn’t appear in the homogeneous equation, so we try a particular solution of the form
\[
y_p = Ae^{-t}.
\]
Substituting in this form, we obtain
\[
-Ae^{-t} + 5Ae^{-t} - 6Ae^{-t} = 4e^{-t}
\]
\[
-2A = 4 \implies A = -2
\]
\[
y(t) = -2e^{-t} + c_1 + c_2e^{-2t} + c_3e^{-3t}.
\]
Solving the initial data, we obtain
\[
y(0) = -2 + c_1 + c_2 + c_3 = 0
\]
\[
\dot{y}(0) = 2 - 2c_2 - 3c_3 = 1
\]
\[
\ddot{y}(0) = -2 + 4c_2 + 9c_3 = 3.
\]
Working with the last two equations, we have
\[
2c_2 + 3c_3 = 1 \implies (-6 + 9)c_3 = -2 + 5
\]
\[
4c_2 + 9c_3 = 5 \implies c_3 = 1.
\]
Substituting this result into either reduced equation, we get $c_2 = -1$. Then using these results in the first equation, we have $-2 + c_1 = 0$. Hence the solution is
\[
y(t) = -2e^{-t} + 2 - e^{-2t} + e^{-3t}.
\]
5. Find the general solution to the differential equation

$$\ddot{y} + \omega^2 y = \cos t, \quad \omega > 0. \quad (4.2)$$

Be sure to account for all $\omega > 0$.

Solution. Substituting in $y = e^{\lambda t}$ into the homogeneous form of (4.2), we have

$$\lambda^2 + \omega^2 = 0 \quad \implies \quad \lambda = \pm i\omega,$$

so

$$y_1 = \cos \omega t, \quad y_2 = \sin \omega t.$$

Then the Wronskian is given by

$$W = \begin{vmatrix} \cos \omega t & \sin \omega t \\ -\omega \sin \omega t & \omega \cos \omega t \end{vmatrix} = \omega (\cos^2 \omega t + \sin^2 \omega t) = \omega.$$

Using the variation of parameters formula, we have

$$y(t) = -\cos \omega t \int \frac{(\sin \omega t)(\cos t)}{\omega} dt + \sin \omega t \int \frac{(\cos \omega t)(\cos t)}{\omega} dt \quad (E)$$

$$= -\cos \omega t \int \frac{\sin(\omega + 1)t + \sin(\omega - 1)t}{\omega} dt + \sin \omega t \int \frac{\cos(\omega - 1)t + \cos(1 + \omega)t}{2\omega} dt$$

$$= \frac{\cos \omega t}{2\omega} \left[ \frac{\cos(1 + \omega)t}{1 + \omega} + \frac{\cos(\omega - 1)t}{\omega - 1} + 2\omega A \right]$$

$$+ \frac{\sin \omega t}{2\omega} \left[ \frac{\sin(\omega - 1)t}{\omega - 1} + \frac{\sin(1 + \omega)t}{1 + \omega} + 2\omega B \right],$$

where the $A$ and $B$ are unknown constants that provide the homogeneous solution. Multiplying out to simplify things, we obtain

$$y(t) = \frac{\cos t}{2\omega(\omega - 1)} + \frac{\cos t}{2\omega(1 + \omega)} = \frac{2\omega \cos t}{2\omega(\omega^2 - 1)} = \frac{\omega t}{\omega^2 - 1}, \quad \omega \neq 1.$$

If $\omega = 1$, we see that (E) becomes

$$y(t) = -\cos t \int \sin t \cos t \, dt + \sin t \int \cos^2 t \, dt$$

$$= -\frac{\cos t}{2} \int \sin 2t \, dt + \frac{\sin t}{2} \int 1 + \cos 2t \, dt$$

$$= \frac{\cos t[\cos 2t + (4A - 1)]}{4} + \frac{\sin t}{2} \left[ t + \frac{\sin 2t}{2} + 2B \right]$$

$$= \frac{t \sin t}{2} + \frac{(4A - 1) + 1}{4} \cos t + B \sin t = \frac{t \sin t}{2} + A \cos t + B \sin t.$$
6. Find the general solution of

\[ \ddot{y} + 2\dot{y} + y = \frac{e^{-t}}{t^2}. \]

**Solution.** We begin by finding the homogeneous solution by substituting in \( e^{\lambda t} \), which yields

\[ \lambda^2 + 2\lambda + 1 = (\lambda + 1)^2 = 0. \]

Since we have a repeated root, the homogeneous solution is given by

\[ y_h(t) = e^{-t}(A + Bt). \]

Then the Wronskian is given by

\[ W = \begin{vmatrix} e^{-t} & te^{-t} \\ -e^{-t} & (1-t)e^{-t} \end{vmatrix} = e^{-2t}. \]

Using the variation of parameters formula, we have that the particular solution is given by

\[ y_p(t) = -e^{-t} \int te^{-t} \frac{e^{-t}}{e^{-2t}} dt + te^{-t} \int e^{-t} \frac{e^{-t}}{e^{-2t}} dt = -e^{-t} \int \frac{dt}{t} + te^{-t} \int \frac{dt}{t^2} = -e^{-t} \log t + e^{-t}. \]

Thus the general solution is given by the homogenous solution plus the particular solution:

\[ y(t) = -e^{-t} \log t + (A + Bt)e^{-t}, \]

where we have folded the \( e^{-t} \) term in the particular solution into the arbitrary constant \( A \).
7. We reconsider the spring system of Homework Set 3, #5. As before, the spring has stiffness $k = 30 \text{ N/m}$, is damped with damping constant $b = 18 \text{ kg/s}$, and is attached to a weight with mass $M = 3 \text{ kg}$. However, now we impose the following discontinuous oscillation of the support (in Newtons; see figure at left):

$$F(t) = \begin{cases} -117 \sin t, & 0 \leq t \leq 2\pi, \\ 0, & t > 2\pi. \end{cases}$$

Initially the spring is extended 1 m and let go.

(a) Solve the resulting system for the displacement $x(t)$ in the region $t \in [0, 2\pi]$.

Solution. From Homework Set 3, we know that the left-hand side of the equation is $3\ddot{x} + 18\dot{x} + 30x$. Then equating this to the additional force given on the support, we have

$$3\ddot{x} + 18\dot{x} + 30x = -117 \sin t, \quad t \in [0, 2\pi]; \quad x(0) = 1, \quad \dot{x}(0) = 0.$$

Using the method of undetermined coefficients, we substitute

$$x_p = p_1 \cos t + p_2 \sin t$$

into our equation to obtain

$$-3(p_1 \cos t + p_2 \sin t) + 18(-p_1 \sin t + p_2 \cos t) + 30(p_1 \cos t + p_2 \sin t) = -117 \sin t$$

$$(27p_1 + 18p_2) \cos t + (27p_2 - 18p_1) \sin t = -117 \sin t$$

$$3p_2 - 2p_1 = -13$$

$$3p_1 + 2p_2 = 0$$

$$p_1 = 2, \quad p_2 = -3.$$

Using our answer to homework set 3, we have that our full solution is given by

$$x(t) = e^{-3t} \left(c_1 \cos t + c_2 \sin t\right) + 2 \cos t - 3 \sin t.$$

Solving the first initial condition, we immediately have that $c_1 + 2 = 1$. Solving the second initial condition, we have

$$e^{-3t} \left(\sin t + c_2 \cos t\right) - 3e^{-3t} \left(-\cos t + c_2 \sin t\right) - 3 \big|_{t=0} = 0$$

$$c_2 = 0$$

$$x(t) = -e^{-3t} \cos t + 2 \cos t - 3 \sin t. \quad \text{(F)}$$

(b) Show that

$$x(2\pi) = -e^{-6\pi} + 2, \quad \dot{x}(2\pi) = 3(e^{-6\pi} - 1). \quad \text{(4.3)}$$

Solution. Substituting $t = 2\pi$ into (F), we obtain

$$x(2\pi) = -e^{-6\pi} + 2,$$
as required. Taking the derivative of (F) and substituting $t = 2\pi$, we obtain
\[
\dot{x}(2\pi) = e^{-3t} \sin t - 3e^{-3t}(-\cos t) - 3 \cos t \bigg|_{t=2\pi} = 3(e^{-6\pi} - 1).
\]

(c) Using the fact that $x$ and $\dot{x}$ should be continuous at $t = 2\pi$, calculate the displacement $x$ in the region $t > 2\pi$.

**Solution.** Since the forcing is now zero, this is exactly Homework Set 3, #5, so our solution is given by
\[
x(t) = e^{-3(t-2\pi)}(d_1 \cos t + d_2 \sin t),
\]
where we have replaced $t$ by $t - 2\pi$ for algebraic convenience. (This is equivalent to multiplying by a constant.) Substituting this expression into (4.3) to obtain the constants, we have that
\[
x(2\pi) = d_1 = 2 - e^{-6\pi},
\]
as required. Taking the derivative of the homogeneous solution and substituting $t = 2\pi$, we obtain
\[
\dot{x}(2\pi) = e^{-3(t-2\pi)}[-(2 - e^{-6\pi}) \sin t + d_2 \cos t]
\]
\[
-3e^{-3(t-2\pi)}[(2 - e^{-6\pi}) \cos t + d_2 \sin t] \bigg|_{t=2\pi}
\]
\[
d_2 - 3(2 - e^{-6\pi}) = 3(e^{-6\pi} - 1)
\]
\[
d_2 = 3 - 6e^{-6\pi}
\]
\[
x(t) = e^{-3(t-2\pi)} [(2 - e^{-6\pi}) \cos t + 3 \sin t].
\]

8. We reconsider the series RLC circuit from #1, but now we attach a battery with voltage $V(t) = E_0 e^{-kt}$ (see figure above right). There is an initial voltage on the capacitor of $V_0$ and initially there is a current $I_0$ in the inductor.

(a) Explain in words why the governing equations for this system are
\[
L \ddot{I} + RI + \frac{I}{C} = -kE_0 e^{-kt}, \quad I(0) = I_0, \quad \dot{I}(0) = \frac{E_0 - V_0 + RI_0}{L}. \quad (4.4)
\]

**Solution.** From notes in class, we have that balance of voltage around the loop yields
\[
L \dot{I} + RI + \frac{1}{C} \int_0^t I + V_0 = E_0 e^{-kt}. \quad (G)
\]
Taking the derivative of this equation with respect to $t$ yields the first equation in (4.4). The initial condition for the current is given in the problem statement. Substituting $t = 0$ into (G), we have
\[
L \dot{I}(0) + RI(0) + V_0 = E_0,
\]
which yields the required equation for $\dot{I}(0)$. 
(b) Solve (4.4) for the steady-state current when \( R = 2, \ C = 1/5, \ L = 1 \). If you take the limit as \( k \to 0 \), do you obtain the same result as you would have for a constant voltage \( E_0 \)?

**Solution.** With the parameters as given, the ODE in (4.4) becomes

\[
\ddot{I} + 2\dot{I} + 5I = -kE_0e^{-kt}.
\]

We have already solved for the homogeneous solution in #1, and it decays as \( t \to \infty \). Hence we don’t need the initial conditions for the steady state, which is just the particular solution.

Using the method of undetermined coefficients, we substitute

\[
I_p(t) = ae^{-kt}
\]

into our equation to obtain

\[
a(k^2 - 2k + 5)e^{-kt} = -kE_0e^{-kt}
\]

\[
a = \frac{-kE_0}{k^2 - 2k + 5}
\]

\[
I_{\text{steady}}(t) = \frac{-kE_0}{k^2 - 2k + 5}.
\]

We note that as \( k \to 0 \), the particular solution becomes zero. If the voltage is a constant \( E_0 \), the initial conditions in (4.4) do not change, but when we take the derivative of the constant voltage to obtain the current equation, we see that the right-hand side would become zero, and hence the particular solution would become zero.

9. Three blends (\( A, B, \) and \( C \)) of alcohol and water are mixed together to form three new mixtures (\( D, E, \) and \( F \)).

Mixture \( D \) is 2 parts \( A \), 2 parts \( B \), 1 part \( C \) and 33 proof (1 proof = 0.5% alcohol)
Mixture \( E \) is 3 parts \( A \), 2 parts \( C \) and 40 proof
Mixture \( F \) is 1 parts \( A \), 3 parts \( B \), 6 part \( C \) and 26 proof

Suppose we want to determine the proof value for \( A, B, \) and \( C \).

(a) Write the necessary system of equations.

**Solution.** We denote the proof level of each mixture by the corresponding letter. If we balance the alcohol level by the fraction of each blend, we have

\[
\frac{2}{5}A + \frac{2}{5}B + \frac{1}{5}C = D = 33,
\]

\[
\frac{3}{5}A + \frac{2}{5}C = E = 40,
\]

\[
\frac{1}{10}A + \frac{3}{10}B + \frac{6}{10}C = F = 26,
\]
where we have to divide the number of parts of each blend by the total number of parts in each mixture to get the volume fraction. Alternatively, if we just want to balance the alcohol level by the parts of each blend, we have

\[
\begin{align*}
2A + 2B + C &= 5D = 165, \\
3A + 2C &= 5E = 200, \\
A + 3B + 6C &= 10F = 260,
\end{align*}
\]

where we have to remember to multiply each proof value by the number of parts it plays in the mixture. Note that the two systems are equivalent upon multiplying by the number of parts in each mixture.

(b) Solve the equations to show that \( A \) is 50 proof, \( B \) is 20 proof, and \( C \) is 25 proof.

**Solution.** Using an augmented matrix form, we have

\[
\begin{align*}
\begin{pmatrix} 2 & 2 & 1 & 165 \\ 3 & 0 & 2 & 200 \\ 1 & 3 & 6 & 260 \end{pmatrix} & \sim \begin{pmatrix} 4 & 0 & -9 & -25 \\ 0 & -4 & -11 & -355 \\ 0 & -9 & -16 & -580 \end{pmatrix} \\
& \sim \begin{pmatrix} 4 & 0 & -9 & -25 \\ 0 & 35 & 0 & 700 \\ 0 & 0 & -35 & -875 \end{pmatrix}.
\end{align*}
\]

Therefore, by back solving we have \( C = 875/35 = 25 \), \( B = 700/35 = 20 \), and \( 4A - 9(25) = -25 \), which implies that \( A = 50 \).

10. Let \( x_s \) be the number of students majoring in CIEG, CPEG, ELEG, or MEEG at the start of a given year, and let \( y_s \) be the number of students not in those majors. Suppose that the following facts are true:

(1) During the course of the year, 70\% of the \( x_s \) majors stay in those majors, and 30\% leave.

(2) During the course of the year, 5\% of the \( y_s \) non-majors change to those majors, and 95\% do not.

Let \( x_e \) and \( y_e \) be the number of majors and non-majors, respectively, at the end of the year.

(a) Write the system of equations needed to solve for \( x_e \) and \( y_e \) as functions of \( x_s \) and \( y_s \).

**Solution.**

\[
\begin{align*}
x_e &= 0.7x_s + 0.05y_s \\
y_e &= 0.3x_s + 0.95y_s
\end{align*}
\]

(H)

(b) If \( y_e = 9,650 \) and \( x_e = 850 \), find \( x_s \) and \( y_s \).
Solution. Using an augmented matrix form, we have

\[
\begin{pmatrix}
a & 0.7 & 0.05 & 850 \\
b & 0.3 & 0.95 & 9650 \\
\end{pmatrix}
\sim
\begin{pmatrix}
19a - b & 13 & 0 & 6500 \\
3a - 7b & 0 & -6.5 & -65000 \\
\end{pmatrix}.
\]

Therefore, by back solving we obtain \( x_s = 6500/13 = 500 \), \( y_s = 65000/6.5 = 10,000 \).

(c) If \( x_s = x_e \) and \( y_s = y_e \), show that the ratio \( y_e/x_e = 6 \).

Solution. Rewriting (H) in this case, we have

\[
\begin{align*}
x_s &= 0.7x_s + 0.05y_s \\
y_s &= 0.3x_s + 0.95y_s \\
\end{align*}
\]

which is a consistent system and has the solution

\[
\frac{y_e}{x_e} = \frac{0.3}{0.05} = 6.
\]