Homework Set 2 Solutions

1. Write the general solution of the equation

\[ \dot{y} = \frac{e^{-t}}{y^2}. \]

*Solution.* Separating variables, and then integrating, we obtain

\[ y^2 \, dy = e^{-t} \, dt \]
\[ \frac{y^3}{3} = -e^{-t} + \frac{C}{3} \]
\[ y^3 = C - e^{-t} \]
\[ y = (C - 3e^{-t})^{1/3}. \]

(Here we divided the constant by 3 initially to get a convenient form, though this can also be done at the end.)

2. Consider the equation

\[ \dot{w} = kt^\alpha \cos^2 w, \quad w(1) = 0, \tag{2.1} \]

where \( k > 0 \) and \( \alpha \) are constants.

(a) Find the solution of (2.1). Be sure to examine the special case when \( \alpha = -1 \).

*Solution.* Separating variables, and then integrating, we obtain

\[ \sec^2 w \, dw = kt^\alpha \, dt \tag{A.1} \]
\[ \tan w = \frac{k t^{\alpha+1}}{\alpha + 1} + C. \tag{A.2} \]

Then using the initial condition, we have

\[ \tan w(1) = \tan 0 = 0 = \frac{k}{\alpha + 1} + C \]
\[ C = -\frac{k}{\alpha + 1} \]
\[ w = \tan^{-1} \left( \frac{k(t^{\alpha+1} - 1)}{\alpha + 1} \right), \quad \alpha \neq -1. \]
If $\alpha = -1$, the solution does not exist because we are dividing by zero in (A.2). Therefore, we substitute $\alpha = -1$ into (A.1) to obtain

\[
\sec^2 w \, dw = \frac{k}{t} \, dt
\]

\[
\tan w = k \log t + C
\]

\[
\tan w(1) = \tan 0 = 0 = C
\]

\[
w = \tan^{-1}(k \log t), \quad \alpha = -1.
\]

(b) Discuss the behavior of the solutions to (2.1) as $t \to \infty$. Remark on the solution for all $\alpha$.

**Solution.** We have that

\[
\lim_{t \to \infty} \frac{kt^{\alpha+1}}{\alpha + 1} = \begin{cases} 
\infty, & \alpha > -1, \\
0, & \alpha < -1,
\end{cases}
\]

\[
\lim_{t \to \infty} k \log t = \infty,
\]

since $k$ is positive. Therefore, we obtain

\[
w(\infty) = \begin{cases} 
\tan^{-1} \infty = \frac{\pi}{2}, & \alpha \geq -1, \\
\tan^{-1} \left( -\frac{k}{\alpha + 1} \right), & \alpha < -1.
\end{cases}
\]

Note the $\geq$ in the first line because the case $\alpha = -1$ behaves like the case $\alpha > -1$.

3. **WITHOUT** solving the problem, determine the interval in $t$ in which the solution of

\[
(t + 2) \dot{y} + y\sqrt{2t + 7} = 3t^2, \quad y(0) = -1
\]

is guaranteed to exist. Is the interval the same if the boundary condition is changed to

\[
y(-3) = 2?
\]

**Solution.** Rewriting our equation in the standard form, we have

\[
\dot{y} = -\frac{\sqrt{2t + 7}}{t + 2} y + \frac{3t^2}{t + 2}.
\]

The second coefficient is undefined for $t = -2$ (from the denominator). The first coefficient is also undefined for $t < -7/2$ (from the square root). If the boundary condition is given at $t = 0 > -2$, the solution is guaranteed to exist for $t > -2$. If the boundary condition is given at $t = -3 < -2$, the solution is guaranteed to exist for $-7/2 < t < -2$.

4. Consider the equation

\[
\dot{y} - ty^3 = 0, \quad y(0) = y_0 > 0.
\]
(a) Write down the solution to the equation.

Solution. Separating variables and then integrating, we have
\[ \frac{dy}{y^3} = t \, dt \]
\[ -\frac{1}{2y^2} = \frac{t^2}{2} - \frac{C}{2} \]
\[ y^{-2} = C - t^2 \]
\[ y^2 = (C - t^2)^{-1}, \]
where we divided \( C \) by \(-2\) at first to make the final expression simpler. Then satisfying
the initial condition, we have
\[ y(0)^2 = y_0^2 = C^{-1} \]
\[ y^2 = (y_0^{-2} - t^2)^{-1} \]
\[ y(t) = (y_0^{-2} - t^2)^{-1/2}. \]

(b) How does the interval of existence for the solution depend on \( y_0 \)?

Solution. The solution exists only for when the quantity under the square root is
positive, so
\[ y_0^{-2} > t^2 \]
\[ |t| < \frac{1}{y_0}. \]

5. Consider the differential equation
\[ 3\ddot{y} + 13\dot{y} + 4y = 0. \]

(a) Find the general solution. Describe the long-time behavior.

Solution. Substituting \( y = e^{\lambda t} \), we obtain
\[ 3\lambda^2 + 13\lambda + 4 = (3\lambda + 1)(\lambda + 4) = 0 \]
\[ \lambda_1 = -1/3, \quad \lambda_2 = -4 \quad \implies \quad y(t) = c_1 e^{-t/3} + c_2 e^{-4t}. \]
Therefore, \( y \to 0 \) as \( t \to \infty \) because both exponents are negative.

(b) Calculate the specific solution for \( y(0) = 4, \dot{y}(0) = -5 \).

Solution. We must solve the following system:
\[ y(0) = c_1 + c_2 = 4 \]
\[ \dot{y}(0) = -\frac{c_1}{3} - 4c_2 = -5 \]
\[ \implies \quad 4c_1 - \frac{c_1}{3} = 4(4) - 5 \quad \implies \quad c_1 = 3, \]
where in the upper right we have combined the equations to eliminate $c_2$. Then using the first equation, we have that $c_2 = 1$, so we have

$$y(t) = 3e^{-t/3} + e^{-4t}.$$ 

6. Write down all equations of the form $a\ddot{y} + b\dot{y} + cy = 0$ such that the solution $y$ approaches a multiple of $e^{-t}$ as $t \to \infty$.

*Solution.* Substituting $y = e^{\lambda t}$, we obtain $a\lambda^2 + b\lambda + c = 0$. If the solution approaches a multiple of $e^{-t}$, we see that the quadratic equation must have $\lambda = -1$ as a root and another root $\lambda_2$ which is *less than* $\lambda = -1$. (Otherwise, the solution would approach $e^{\lambda_2 t}$.) So we must have

$$a(\lambda + 1)(\lambda - \lambda_2) = 0,$$

$$a\lambda^2 + a(1 - \lambda_2)\lambda - a\lambda_2 = 0, \quad \lambda_2 < -1.$$

7. Consider the following system of coupled first-order ODEs:

$$3\dot{x} + x + 2\dot{y} + 5y = 0, \quad (2.2a)$$
$$-2x + \dot{y} + 4y = 0. \quad (2.2b)$$

(a) Eliminate $x$ from the system to obtain a second-order ODE for $y$.

*Solution.* We take twice (2.2a) and add it to thrice the derivative of (2.2b) to obtain

$$2(3\dot{x} + x + 2\dot{y} + 5y) + 3(-2\dot{x} + \dot{y} + 4y) = 2x + 4\dot{y} + 10y + 3\ddot{y} + 12\dot{y} = 0.$$

Then adding (2.2b) to the above, we have

$$\dot{y} + 4y + 10y + 3\ddot{y} + 16\dot{y} = 0,$$
$$3\ddot{y} + 17\dot{y} + 14y = 0.$$

(b) Show that the general solution for $y$ is

$$y(t) = c_1 e^{-14t/3} + c_2 e^{-t},$$

and find the corresponding general solution for $x$.

*Solution.* Substituting $y = e^{\lambda t}$, we obtain

$$3\lambda^2 + 17\lambda + 14 = 0,$$

$$(3\lambda + 14)(\lambda + 1) = 0 \quad \Rightarrow \quad \lambda_1 = -\frac{14}{3}, \quad \lambda_2 = -1$$

$$y(t) = c_1 e^{-14t/3} + c_2 e^{-t}.$$  

Then substituting this result into (2.2b) to obtain $x$, we have

$$2x = \dot{y} + 4y = c_1 \left(-\frac{14}{3} + 4\right) e^{-14t/3} + c_2 (-1 + 4)e^{-t}$$

$$x = -\frac{c_1}{3} e^{-14t/3} + \frac{3c_2}{2} e^{-t}.$$
8. For the equation
\[ 2\ddot{y} + 5\dot{y} - 3y = 0, \]
find the fundamental set \( \{y_1(t), y_2(t)\} \) where
\[ y_1(0) = 1, \quad \dot{y}_1(0) = 0; \quad y_2(0) = 0, \quad \dot{y}_2(0) = 1. \]

**Solution.** Substituting \( y = e^{\lambda t} \), we obtain
\[ 2\lambda^2 + 5\lambda - 3 = 0 \]
\[ (2\lambda - 1)(\lambda + 3) = 0 \quad \implies \quad \lambda_1 = \frac{1}{2}, \quad \lambda_2 = -3, \]
so solutions are of the form \( y = c_1 e^{t/2} + c_2 e^{-3t} \). Therefore, for \( y_1 \) we must solve
\[ c_1 + c_2 = 1 \quad \implies \quad 3c_1 + \frac{c_1}{2} = 3(1) + 0 \]
\[ c_1 = \frac{6}{7} \]
Then substituting this result into the first equation, we have
\[ y_1(t) = \frac{6e^{t/2} + e^{-3t}}{7}. \]
For \( y_2 \) we must solve
\[ c_1 + c_2 = 0 \quad \implies \quad 3c_1 + \frac{c_1}{2} = 3(0) + 1 \]
\[ c_1 = \frac{2}{7} \]
Then substituting this result into the first equation, we have
\[ y_2(t) = \frac{2e^{t/2} - 2e^{-3t}}{7}. \]

9. Consider the equation
\[ (t^2 - 1)\ddot{y} + t\dot{y} + \frac{3y}{\cos t} = 0. \]
Find all intervals where this equation is guaranteed to have a unique solution.
(Consider \( t \) to be of either sign.)

**Solution.** Rewriting the equation in standard form, we have
\[ \ddot{y} + \frac{t}{t^2 - 1} \dot{y} + \frac{3y}{(t + 1)(t - 1)\cos t} = 0. \]
The coefficient of \( y \) is undefined whenever \( t = 1, \ t = -1, \) and \( t = (2n + 1)\pi/2 \), where \( n \) is an integer. Therefore, the equation has a unique solution in any interval not containing those points.

10. Consider the ODE
\[
y^{(3)} - 13y - 12y = 0. \tag{2.3}
\]

(a) Show by direct substitution that three solutions of (2.3) are given by \( \{e^{-t}, e^{-3t}, e^{4t}\} \).

*Solution.* Plugging in each of the functions, we have
\[
\begin{align*}
\frac{d^3(e^{-t})}{dt^3} - 13\frac{d(e^{-t})}{dt} - 12e^{-t} &= (-1 + 13 - 12)e^{-t} = 0, \\
\frac{d^3(e^{-3t})}{dt^3} - 13\frac{d(e^{-3t})}{dt} - 12e^{-3t} &= (-27 + 39 - 12)e^{-3t} = 0, \\
\frac{d^3(e^{4t})}{dt^3} - 13\frac{d(e^{4t})}{dt} - 12e^{4t} &= (64 - 52 - 12)e^{4t} = 0.
\end{align*}
\]

(b) Show that the Wronskian of these three solutions is constant.

*Solution.* Here is one solution:
\[
W\{e^{-t}, e^{-3t}, e^{4t}\} = 
\begin{vmatrix}
e^{-t} & e^{-3t} & e^{4t} \\
-e^{-t} & -3e^{-3t} & 4e^{4t} \\
e^{-t} & 9e^{-3t} & 16e^{4t}
\end{vmatrix} = e^{-t}e^{-3t}e^{4t} \begin{vmatrix} 1 & 1 & 1 \\ -1 & -3 & 4 \\ 1 & 9 & 16 \end{vmatrix} \\
= (-48 - 36) - (-16 - 4) + (-9 + 3) = -70.
\]