Homework Set 1 Solutions

1. Determine by inspection two solutions of

\[ y'' + y = 0. \]

\[(1.1)\]

**Solution.** We recall from calculus that the trigonometric function \( \sin x \) has the property that

\[ \frac{d^2(\sin x)}{dx^2} = -\sin x, \]

and hence \( y = \sin x \) satisfies (1.1). Similarly, since

\[ \frac{d^2(\cos x)}{dx^2} = -\cos x, \]

we see that \( y = \cos x \) is another solution of (1.1).

2. Consider the driven \( RC \) circuit shown above left. The current through the resistor is given by \( I_R = V/R \), where \( V \) is the voltage, and the current through the capacitor is given by \( I_C = CV' \). Let the driving current \( I = B \sin \omega t \), where \( B \) is the constant amplitude and \( \omega \) the constant frequency. (The directions of the currents are indicated by the arrows.) The sum of the currents into the node above the resistor must be zero; use this fact to obtain the differential equation

\[ V' + \frac{1}{RC} V = \frac{B \sin \omega t}{C}. \]
Solution. By the directions of the arrows, the current flowing out of the node is given by $I_R + I_C$, while the current flowing into the node is the imposed current $B \sin \omega t$. Since current in must equal out (so the net current is zero), we have that

$$C\dot{V} + \frac{1}{R}V = B \sin \omega t,$$

and dividing by $C$ gives the desired result.

3. Consider the driven $RL$ circuit shown above right. The voltage through the inductor is given by $V_L = L\dot{I}$, where $I$ is the current. Let the driving voltage $V = A \sin \omega t$, where $A$ is the constant amplitude and $\omega$ the constant frequency. The sum of the voltage around this loop must be zero; use this fact to obtain the differential equation

$$\dot{I} + \frac{R}{L}I = \frac{A \sin \omega t}{L}.$$

Solution. Since the sign of the voltage drop is the same as the sign of the first sign as the current goes around, we see that the imposed voltage is negative. Therefore, we have $V_L + V_R - A \sin \omega t = 0$. From problem #2, we see that $V_R = IR$, so we have

$$L\dot{I} + RI = A \sin \omega t,$$

from which the desired result immediately follows.

4. Consider the equation

$$6\dot{y} = 3y - 2. \quad (1.2)$$

(a) Show by direct substitution that a solution of the form

$$y(t) = A + Be^{\lambda t}. \quad (1.3)$$

satisfies (1.2). Calculate exact values for as many of the constants $\{A, B, \lambda\}$ as possible. Which terms in (1.2) help you determine which constants?

Solution. Substituting (1.3) into (1.2), we obtain

$$6(\lambda Be^{\lambda t}) = 3(A + Be^{\lambda t}) - 2$$

$$3B(2\lambda - 1)e^{\lambda t} = 3A - 2.$$

Matching coefficients, we see that (1.3) is a solution as long as $\lambda = 1/2$ and $A = 2/3$. We are unable to determine $B$ at this time. The $6\dot{y}$ and $3\dot{y}$ terms determined $\lambda$, and the right-hand side of (1.2) determined $A$.

Now suppose that in addition to (1.2),

$$y(0) = 0. \quad (1.4)$$

(b) Use (1.4) to calculate exact values for all the constants in the proposed solution (1.3).
**Solution.** From part (a), we have that our solution is

\[ y(t) = \frac{2}{3} + Be^{t/2}. \]

Substituting \( t = 0 \) into the above, we have

\[ y(0) = \frac{2}{3} + B = 0, \]

so \( B = -2/3 \) and our final solution is

\[ y(t) = \frac{2 - 2e^{t/2}}{3}. \]

5. Suppose that a mass on a spring is driven by some force \( F(t) \). The equation for \( x(t) \), the position of the mass, is given by

\[ \ddot{x} + kx = F(t), \quad (1.5) \]

where \( k \) is Hooke’s constant. Rewrite (1.5) as a set of two first-order equations, and interpret physically the new variable you must introduce when doing so.

**Solution.** If we let \( v = \dot{x} \), then (1.5) becomes \( \dot{v} + kx = F(t) \). Rewriting this as a system, we have

\[ \begin{align*}
\dot{x} &= v, \\
\dot{v} &= F(t) - kx.
\end{align*} \]

Here \( v \) is just the velocity.

6. Consider the differential equation

\[ \dot{y} - 7y = e^{-2t}, \quad y(0) = y_0. \]

(a) Find the solution for any constant \( y_0 \).

**Solution.** Since \( p(t) = -7 \), the integrating factor is \( e^{-7t} \). Multiplying by this factor and integrating, we have

\[ e^{-7t} \dot{y} - 7e^{-7t}y = e^{-9t} \]

\[ \frac{d(e^{-7t}y)}{dt} = e^{-9t} \]

\[ e^{-7t}y = -\frac{e^{-9t}}{9} + C \]

\[ y(t) = -\frac{e^{-2t}}{9} + Ce^{7t} \]

\[ y(0) = C - \frac{1}{9} = y_0 \]

\[ C = y_0 + \frac{1}{9} \]

\[ y(t) = -\frac{e^{-2t}}{9} + \left(y_0 + \frac{1}{9}\right) e^{7t}. \]
(b) Describe how the long-time behavior of $y$ varies with $y_0$. (In other words, does the solution decay, tend to positive or negative infinity, etc.)

Solution. As $t \to \infty$, $y(t)$ becomes exponentially large, and the sign of $y(t)$ is the same as the sign of $y_0 + 1/9$. Therefore, we have

$$\lim_{t \to \infty} y(t) = \begin{cases} \infty, & y_0 > -1/9, \\ -\infty, & y_0 < -1/9. \end{cases}$$

(c) Find the critical value of $y_0$ which separates the two types of behaviors.

Solution. From part (b), we see that the critical value is $y_0 = -1/9$.

(d) Describe the long-time behavior of $y$ for that specific value of $y_0$.

Solution. For $y_0 = -1/9$, the solution is $-e^{-2t}/9$, which goes to 0 as $t \to \infty$.

7. Show (by deriving the solution, NOT by direct substitution) that the solution to the differential equation

$$t^2y - 2(2t^2 - t)y = e^{4t}, \quad y(1) = 0$$

is given by

$$y(t) = \left(\frac{1}{t} - \frac{1}{t^2}\right) e^{4t}.$$

Solution. Dividing by $t^2$ to obtain the standard form, we have

$$\dot{y} - 2\left(2 - \frac{1}{t}\right)y = \frac{e^{4t}}{t^2},$$

so the integrating factor is

$$\exp\left(-2\int 2 - \frac{1}{t} \, dt\right) = \exp(-2(2t - \log t)) = t^2 e^{-4t}.$$ 

Multiplying and integrating, we have

$$\frac{d}{dt} \left(t^2 e^{-4t} y\right) = 1$$

$$y(t) = t^{-2} e^{4t} (t + C) = e^{4t} \left(\frac{1}{t} + \frac{C}{t^2}\right)$$

$$y(1) = e^4 (1 + C) = 0$$

$$C = -1$$

$$y(t) = e^{4t} \left(\frac{1}{t} - \frac{1}{t^2}\right).$$

8. Find the general solution of the differential equation

$$\dot{y} + (\sin t) y = \sin t.$$
Solution. Since \( p(t) = \sin t \), the integrating factor is

\[
\exp \left( \int \sin t \, dt \right) = e^{-\cos t}.
\]

Multiplying and integrating, we have

\[
\frac{d}{dt} \left( e^{-\cos t} y \right) = \sin t e^{-\cos t} \]

\[
y(t) = e^{\cos t} \left( e^{-\cos t} + C \right) = 1 + Ce^{\cos t}
\]

9. In #2, you showed that for the driven RC circuit shown in the figure, the governing equation for the current \( V \) is given by

\[
\dot{V} + \frac{1}{RC} V = \frac{B \sin \omega t}{C}.
\]

(a) If the initial voltage is zero, \( R = 2 \Omega \), \( B = 1/2 \) A, and \( C = 1 \) F, find the voltage in the circuit as a function of time. Do the initial conditions matter as \( t \to \infty \)?

Solution. Making these substitutions, we have

\[
\dot{V} + \frac{1}{2} V = 2 \sin \omega t, \quad V(0) = 0
\]

\[
\frac{d}{dt} \left( e^{t/2} V \right) = \frac{1}{2} e^{t/2} \sin \omega t
\]

\[
e^{t/2} V = \frac{K_s}{2} + k, \quad K_s \equiv \int e^{t/2} \sin \omega t \, dt,
\]

where \( k \) is a constant. To calculate \( K_s \) we integrate by parts twice to obtain

\[
K_s = 2e^{t/2} \sin \omega t - 2\omega \int e^{t/2} \cos \omega t \, dt
\]

\[
= 2e^{t/2} \sin \omega t - 2\omega \left( 2e^{t/2} \cos \omega t + 2\omega \int e^{t/2} \sin \omega t \, dt \right)
\]

\[
= 2e^{t/2} \sin \omega t - 4\omega e^{t/2} \cos \omega t - 4\omega^2 K_s,
\]
where we have used the definition of $K_s$ in (A). Continuing to simplify, we have

$$
(1 + 4\omega^2)K_s = 2e^{t/2} \sin \omega t - 4\omega e^{t/2} \cos \omega t
$$

$$
K_s = \frac{2e^{t/2}(\sin \omega t - 2\omega \cos \omega t)}{1 + 4\omega^2}
$$

$$
V = e^{-t/2} \left( \frac{K_s}{2} + k \right) = \frac{\sin \omega t - 2\omega \cos \omega t}{1 + 4\omega^2} + ke^{-t/2},
$$

where $k$ is a constant we will use to solve the initial data. But note that for any $k$, as $t \to \infty$ that term goes to zero while the others oscillate. So the initial conditions don’t make any difference to the steady state the solution approaches as $t \to \infty$.

Satisfying the initial condition, we have

$$
V(0) = 0 = \frac{-2\omega}{1 + 4\omega^2} + k
$$

$$
k = \frac{2\omega}{1 + 4\omega^2},
$$

$$
V(t) = \frac{\sin \omega t - 2\omega (\cos \omega t - e^{-t/2})}{1 + 4\omega^2}.
$$

(B)

Often when studying electrical circuits the imposed current or voltage is turned on or off suddenly. In the equation

\[ \dot{y} + p(t)y = g(t), \]

this corresponds to $g(t)$ being discontinuous. When this occurs, we solve the problem in each interval where $g$ is continuous, and then require that $y$ be continuous where the intervals join together.

Consider the circuit above, but now suppose that we shut the current off after one cycle, so we have

\[ I = \begin{cases} 
B \sin \omega t, & 0 \leq t \leq 2\pi/\omega, \\
0, & t > 2\pi/\omega.
\end{cases} \]

(b) Using your answer to (a) (with $B = 1/2$ A), calculate $V(2\pi/\omega)$.

Solution. Using (B), we have

$$
V(2\pi/\omega) = \frac{-2\omega(1 - e^{-\pi/\omega})}{1 + 4\omega^2}.
$$

(c) Using your answer to (b) as an “initial” condition, solve the differential equation for $t > 2\pi/\omega$.

Solution. When the voltage has been turned off, the resulting equation is

\[ \dot{V} + \frac{1}{2} V = 0, \]
the solution of which is trivially $V = ke^{-t/2}$. Solving for $k$, we have

$$V(2\pi/\omega) = ke^{-\pi/\omega}$$

$$k = V(2\pi/\omega)e^{\pi/\omega}$$

$$V(t) = \frac{-2\omega(1 - e^{-\pi/\omega})}{1 + 4\omega^2} e^{-(t/2 - \pi/\omega)}, \quad t > 2\pi/\omega.$$  

10. Consider the differential equation

$$\dot{y} - \frac{\alpha}{t} y = \frac{1}{t}.$$  

(a) Find the general solution for any $\alpha$. Be sure to consider the special case where $\alpha = 0$.

Solution. The integrating factor is

$$\exp\left(-\alpha \int \frac{1}{t} \, dt\right) = \exp(-\alpha \log t) = t^{-\alpha},$$

so we have

$$\frac{d}{dt} (t^{-\alpha} y) = t^{-(\alpha + 1)}$$

$$y = t^\alpha \left( \frac{t^{-\alpha}}{-\alpha} + C \right) = -\frac{1}{\alpha} + Ct^\alpha, \quad \alpha \neq 0.$$  

If $\alpha = 0$, we have

$$\dot{y} = \frac{1}{t} \quad \implies \quad y = \log t + C.$$  

(b) For what values of $\alpha$ does the solution stay bounded as $t \to \infty$?

Solution. Looking at our solution, we see that for $\alpha < 0$, the solution decays as $t \to \infty$. 

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