Homework Set 8 (Revised)

Read sections P4.1, P4.3, and P6.3.

Section P6.3

1. Consider the two ordered bases $B = [2, 3 + x, 1 - x^2]$ and $C = [1, 2 - x, x^2 + x + 2]$ for $P_2$.
   (a) Find the transition matrix $C \leftarrow B$.
   (b) Calculate $b = [x^2 - 1]_B$ and $c = [x^2 - 1]_C$.
   (c) Verify that $c = C \leftarrow B b$.

2. Consider the matrix and vectors $B = \begin{pmatrix} 14 & 15 \\ -10 & -11 \end{pmatrix}$, $u_1 = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$, $u_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$.
   (a) Verify that $B u_1 = 4 u_1$ and $B u_2 = -u_2$.
   (b) Find the transition matrix $T = U \leftarrow E$ corresponding to the change of basis from the standard basis $E = [e_1, e_2]$ to the basis $U = [u_1, u_2]$.
   (c) Show that the $U$-coordinates of $(3, -1)^T$ are $(2, 3)^T$.
   (d) Calculate $T B T^{-1}$.

Section P4.1

3. Determine which of the vectors $k_1 = \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix}$, $k_2 = \begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix}$, $k_3 = \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}$ are eigenvectors for $A = \begin{pmatrix} -1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 3 & -1 \end{pmatrix}$.
   For any eigenvectors, find the corresponding eigenvalue.

4. Let $A$ be an $n \times n$ matrix and let $B = A - \alpha I$. If $A z = \lambda z$, show that $z$ is also an eigenvector for $B$. What is its eigenvalue with respect to $B$?
5. Consider the matrix

\[ A = \begin{pmatrix} 5 & 0 \\ 2 & 1 \end{pmatrix}. \]

(a) Find an eigenvector for \( A \) corresponding to \( \lambda = 1 \).
(b) Find an eigenvector for \( A \) corresponding to \( \lambda = 5 \).

**Section P4.3**

6. Let \( A \) be a matrix whose rows all add up to the same constant \( \delta \). Show that \( \delta \) is an eigenvalue of \( A \).

7. Find the eigenvalues and corresponding eigenspaces for the following matrices:

\[ A_1 = \begin{pmatrix} 6 & 1 \\ -4 & 6 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 5 & 4 \\ 1 & 2 \end{pmatrix}. \]

8. Consider the following matrix:

\[ C = \begin{pmatrix} -2 & 2 & 0 \\ 1 & -2 & 1 \\ 0 & 2 & -2 \end{pmatrix}. \]

(Matrices of this form occur regularly in physical problems. For instance, this matrix could represent the diffusion of heat in a one-dimensional bar.) Find the eigenvalues and a basis for each of the corresponding eigenspaces.

9. (BH) Consider the following matrix:

\[ F = \begin{pmatrix} 1 & 3 & -2 \\ -2 & 0 & 4 \\ 3 & 5 & -6 \end{pmatrix}. \]

We wish to calculate the eigenvalues of this matrix without using the characteristic polynomial.

(a) Use facts about determinants to explain why \( \lambda_1 = 0 \).
(b) Use your answer to #6 to obtain \( \lambda_2 \).
(c) Use facts about the trace to determine the third eigenvalue.
10. The *Cayley-Hamilton theorem* states that every matrix $A \in \mathbb{R}^{n \times n}$ is a root of its characteristic polynomial. So if

$$p_A(\lambda) = a_0 + \sum_{j=1}^{n} a_j \lambda^j$$

is the characteristic polynomial of $A$, then

$$p_A(A) = a_0 I + \sum_{j=1}^{n} a_j A^j = O.$$ 

Verify the Cayley-Hamilton Theorem for the matrix

$$A = \begin{pmatrix} 3 & 2 \\ 7 & 4 \end{pmatrix}. \quad (8.1)$$