Homework Set 2

Read sections Z1.1, Z1.2, Z2.2, Z3.1.

Sections Z1.2, Z2.2

1. Write the general solution of the equation

\[ \dot{y} = \frac{e^{-t}}{y^2}. \]

2. Consider the equation

\[ \ddot{w} = kt^\alpha \cos^2 w, \quad w(1) = 0, \quad w(1) = 0, \]  

where \( k > 0 \) and \( \alpha \) are constants.

(a) Find the solution of (2.1). Be sure to examine the special case when \( \alpha = -1 \).

(b) Discuss the behavior of the solutions to (2.1) as \( t \to \infty \). Remark on the solution for all \( \alpha \).

3. WITHOUT solving the problem, determine the interval in \( t \) in which the solution of

\[ (t + 2) \dot{y} + y\sqrt{2t + 7} = 3t^2, \quad y(0) = -1 \]

is guaranteed to exist. Is the interval the same if the boundary condition is changed to

\[ y(-3) = 2? \]

4. Consider the equation

\[ \dot{y} - ty^3 = 0, \quad y(0) = y_0 > 0. \]

(a) Write down the solution to the equation.

(b) How does the interval of existence for the solution depend on \( y_0 \)?
Section Z3.1

5. Consider the differential equation

\[ 3\ddot{y} + 13\dot{y} + 4y = 0. \]

(a) Find the general solution. Describe the long-time behavior.
(b) Calculate the specific solution for \( y(0) = 4, \dot{y}(0) = -5 \).

6. Write down all equations of the form \( a\ddot{y} + b\dot{y} + cy = 0 \) such that the solution \( y \) approaches a multiple of \( e^{-t} \) as \( t \to \infty \).

7. Consider the following system of coupled first-order ODEs:

\begin{align*}
3\dot{x} + x + 2\dot{y} + 5y &= 0, \quad (2.2a) \\
-2x + \dot{y} + 4y &= 0. \quad (2.2b)
\end{align*}

(a) Eliminate \( x \) from the system to obtain a second-order ODE for \( y \).
(b) Show that the general solution for \( y \) is

\[ y(t) = c_1 e^{-14t/3} + c_2 e^{-t}, \]

and find the corresponding general solution for \( x \).

8. For the equation

\[ 2\ddot{y} + 5\dot{y} - 3y = 0, \]

find the fundamental set \( \{y_1(t), y_2(t)\} \) where

\[ y_1(0) = 1, \quad \dot{y}_1(0) = 0; \quad y_2(0) = 0, \quad \dot{y}_2(0) = 1. \]

9. Consider the equation

\[ (t^2 - 1)\ddot{y} + t\dot{y} + \frac{3y}{\cos t} = 0. \]

Find all intervals where this equation is guaranteed to have a unique solution.
(Consider \( t \) to be of either sign.)

10. Consider the ODE

\[ y^{(3)} - 13\dot{y} - 12y = 0. \quad (2.3) \]

(a) Show by direct substitution that three solutions of (2.3) are given by \( \{e^{-t}, e^{-3t}, e^{4t}\} \).
(b) Show that the Wronskian of these three solutions is constant.