Beats

In class we considered the solution of the system

\[ \ddot{u} + k^2 u = F_0 \cos \omega t, \quad u(0) = 0, \quad \dot{u}(0) = 0, \]

and found the answer to be

\[ u = \frac{2F_0}{k^2 - \omega^2} \sin \left( \frac{(k - \omega)t}{2} \right) \sin \left( \frac{(k + \omega)t}{2} \right), \quad k \neq \omega. \]

We consider the first two factors to be a time-varying amplitude (or envelope). If \( k = \omega \), the envelope grows linearly and we have

\[ u = \frac{F_0 t}{2k} \sin kt. \]

Now take \( k = 2 \) and \( F_0 = 1 \). Here are some graphs of the solution for various values of \( \omega \).

\[ u(t) \text{ vs. } t \text{ for } \omega = 3. \text{ Thin curve: envelope. Thick curve: solution.} \]

In this case the forcing frequency is far from the natural frequency \( \lambda = 2 \), so we have languid oscillations within the beats.
As the forcing frequency nears the natural frequency $k = 2$, we see much more rapid oscillations within the beats, but note that the envelope eventually reaches a maximum and will return to zero.

If the forcing frequency is the same as the natural frequency, the envelope increases linearly and so do the amplitude of the oscillations.