

## Homework Set 9 (Revised)

Read sections 4.2 and 6.6.

### Section 6.6

1. (BH) Let  $\mathcal{L} : \mathcal{R}^3 \rightarrow \mathcal{R}^3$  be defined by

$$\mathcal{L}(\mathbf{e}_1) = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}, \quad \mathcal{L}(\mathbf{e}_2) = \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}, \quad \mathcal{L}(\mathbf{e}_3) = \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}.$$

Let  $T = \{\mathcal{L}(\mathbf{e}_1), \mathcal{L}(\mathbf{e}_2), \mathcal{L}(\mathbf{e}_3)\}$  be an ordered basis for  $\mathcal{R}^3$  and  $S$  be the standard basis for  $\mathcal{R}^3$ .

- (a) Find the matrix representation of  $\mathcal{L}$  with respect to  $S$  and  $T$ . Call it  ${}_T M_S$ .  
(b) Find  $\mathcal{L}((2, 1, -1)^T)$  using the matrix obtained in part (a). Express your answer in both  $S$ - and  $T$ -coordinates.
2. (BH) Let  $\mathcal{L} : \mathcal{P}_2 \rightarrow \mathcal{P}_1$  be defined by  $\mathcal{L}(p(t)) = p'(t) + p(0)$ . Consider the ordered bases  $S = \{t^2, t, 1\}$  and  $S' = \{t^2 - 1, t + 1, t\}$  for  $\mathcal{P}_2$  and the ordered bases  $T = \{t, 1\}$  and  $T' = \{2, 1 - t\}$  for  $\mathcal{P}_1$ . Find the matrix representation of  $\mathcal{L}$  with respect to
- (a)  $S$  and  $T$ .  
(b)  $S'$  and  $T'$ .  
(c) Find  $\mathcal{L}(t^2 + 3t + 1)$  using the definition of  $\mathcal{L}$  and the matrices obtained in parts (a) and (b).
3. (BH) Write the standard matrix for each of the following transformations (HW 8, #9):
- (a)  $L : \mathcal{R}^3 \rightarrow \mathcal{R}^3$  projects every point  $\mathbf{x}$  to the plane  $x_1 = x_2$ . (This is an orthogonal projection.)  
(b)  $L : \mathcal{R}^2 \rightarrow \mathcal{R}^2$  reflects a vector about the  $x_2$ -axis and then rotates it by  $\pi/4$ .

### Section 4.2

4. (BH)  
(a) Evaluate

$$\begin{vmatrix} t+2 & 2 \\ 2 & t-1 \end{vmatrix}.$$

- (b) Find values of  $t$  such that

$$\begin{vmatrix} t+2 & 2 \\ 2 & t-1 \end{vmatrix} = 0.$$

5. (MP)

(a) Evaluate

$$\begin{vmatrix} -(t+5) & 6 & 1 \\ -1 & 2-t & 1 \\ -8 & 6 & 4-t \end{vmatrix}.$$

(b) Find values of  $t$  such that

$$\begin{vmatrix} -(t+5) & 6 & 1 \\ -1 & 2-t & 1 \\ -8 & 6 & 4-t \end{vmatrix} = 0.$$

6. (BH) Consider the following matrix in  $\mathcal{R}^{5 \times 5}$ :

$$A = \begin{pmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 \end{pmatrix},$$

where the \*s are arbitrary entries. Show that  $\det A = 0$ .7. (BH) Calculate  $\det B$ , where

$$B = \begin{pmatrix} 0 & 0 & 2 & 4 \\ 2 & 3 & 5 & -1 \\ 0 & 0 & -1 & 2 \\ 2 & 0 & 1 & 1 \end{pmatrix}.$$

*(Hint: At each stage, pick the row or column that produces the easiest expressions.)*

8.

(a) (MP) Verify that  $\det(AB) = (\det A)(\det B)$  for the following matrices:

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 2 \\ 3 & 4 & 8 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & 4 & 2 \\ 2 & -1 & 3 \\ -1 & 3 & 1 \end{pmatrix}.$$

(b) (BH) Verify that  $\det(AB) = (\det A)(\det B)$  for the following matrices:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & 3 \\ 0 & 0 & -7 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 0 & 0 \\ 4 & 1 & 0 \\ 3 & 2 & 2 \end{pmatrix}.$$

9. (BH) Suppose that  $A$ ,  $B$ , and  $C$  are  $n \times n$  matrices such that  $\det A = 2$ ,  $\det B = 0$ , and  $\det C = -5$ . Calculate the following determinants:
- (a)  $\det C^{-1}$
  - (b)  $\det(4A)$
  - (c)  $\det(ABC)$
  - (d)  $\det(C^T A^{-1})$
10. (BH) Consider the following *Vandermonde* matrix:

$$A = \begin{pmatrix} 1 & a_1 & a_1^2 & \cdots & a_1^{n-1} \\ 1 & a_2 & a_2^2 & \cdots & a_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & a_n & a_n^2 & \cdots & a_n^{n-1} \end{pmatrix}$$

- (a) Explain why  $\det A = 0$  if  $a_i = a_j$  for any  $i \neq j$ .
- (b) What is the highest power of  $a_i$  found in any term of  $\det A$ ?
- (c) Show that

$$\det A = C \prod_{1 \leq i < j \leq n} (a_j - a_i) \tag{9.1}$$

for some constant  $C$ . (Actually, the constant is 1.)