Homework Set 3 Solutions

1. (BH) Solve the linear system

\[ \begin{align*}
    x - y + z &= 0 \\
    -x + 3y + z &= 5 \\
    3x + y + 7z &= 2.
\end{align*} \]

**Solution.** We form the augmented matrix and then row reduce:

\[
\begin{pmatrix}
a & -1 & 1 & 0 \\
b & 3 & 1 & 5 \\
c & 3 & 1 & 7 & 2
\end{pmatrix}
\sim
\begin{pmatrix}
a & -b & 2 & -4 & 0 & -5 \\
e & a & b & 0 & 2 & 2 & 5 \\
f & 3b + c & 0 & 10 & 10 & 17
\end{pmatrix}
\sim
\begin{pmatrix}
d & 2 & -4 & 0 & -5 \\
e & 0 & 2 & 2 & 5 \\
f & 5e - f & 0 & 0 & 8
\end{pmatrix}.
\]

The last line corresponds to the inconsistent equation \(0 = 8\), so the system has no solution.

2. (BH) Let

\[
A = \begin{pmatrix}
1 & -2 & 0 & 2 \\
2 & -3 & -1 & 5 \\
1 & 3 & 2 & 5 \\
1 & 1 & 0 & 2
\end{pmatrix}.
\]

(a) Find matrices \(B\) and \(C\) in row echelon form that are row equivalent to \(A\).

**Solution.** (There are many correct answers to this problem.)

\[
\begin{align*}
    a \begin{pmatrix}
        1 & -2 & 0 & 2 \\
        2 & -3 & -1 & 5 \\
        1 & 3 & 2 & 5 \\
        1 & 1 & 0 & 2
    \end{pmatrix}
    & \sim
    d = a - b \begin{pmatrix}
        2 & -4 & 0 & -5 \\
        0 & 2 & 2 & 5 \\
        0 & 10 & 10 & 17
    \end{pmatrix}
    \sim
    d \begin{pmatrix}
        2 & -4 & 0 & -5 \\
        0 & 2 & 2 & 5 \\
        5e - f & 0 & 0 & 8
    \end{pmatrix}.
\end{align*}
\]

Once we have done these preliminary steps, we see that a matrix \(B\) can be found simply by dividing the last row by \(-15\):

\[
B = \begin{pmatrix}
1 & -2 & 0 & 2 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 4 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]
Subtracting 4 times the last row from the third row and twice the last row from the first row yields another matrix $C$:

$$C = \begin{pmatrix} 1 & -2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

(b) Find a matrix $D$ in reduced row echelon form that is row equivalent to $A$.

*Solution.* (There is only one correct answer to this part.) Adding twice the second row of $C$ to the first row yields $D$:

$$D = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

3. (MP) Solve the linear system with the given augmented matrix, or show it has no solution.

(a): \[
\begin{pmatrix} 1 & 2 & 3 & 1 & | & 8 \\ 1 & 3 & 0 & 1 & | & 7 \\ 1 & 0 & 2 & 1 & | & 3 \end{pmatrix}
\]

(b): \[
\begin{pmatrix} 1 & 1 & 3 & -3 & | & 0 \\ 0 & 2 & 1 & -3 & | & 3 \\ 1 & 0 & 2 & -1 & | & -1 \end{pmatrix}
\]

4. (BH) Consider the following system:

$$2x_1 + ax_2 = 3, \quad (3.1a)$$

$$x_1 + x_2 = b. \quad (3.1b)$$

(a) Write (3.1) as an augmented matrix.

*Solution.* \[
\begin{pmatrix} 2 & a & 3 \\ 1 & 1 & b \end{pmatrix}
\]

(b) Write your answer to (a) in reduced row echelon form.

*Solution.*

\[
a \begin{pmatrix} 2 & a & 3 \\ 1 & 1 & b \end{pmatrix} \sim \begin{pmatrix} 2 - a & 0 & 3 - ab \\ a - 2b & a - 2 & 3 - 2b \end{pmatrix}
\]
\[
\begin{pmatrix}
c/(2-a) & 1 & 0 & (3-ab)/(2-a) \\
d/(a-2) & 0 & 1 & (3-2b)/(a-2)
\end{pmatrix}
\]

(c) For which value(s) of \(a \) and \(b\) will (3.1) have
(i) more than one solution?
(ii) exactly one solution?
(iii) no solution?

Solution. Clearly if \(a \neq 2\), all of the entries in the matrix are bounded and the problem has exactly one solution. If \(a = 2\), the denominators of the last entries are zero, and hence there is no solution except possibly when the numerator is also zero. The numerator is zero when \(b = 3/2\), so there is no solution for \(a = 2, b \neq 3/2\). When \(a = 2\) and \(b = 3/2\), the two equations are redundant, and so there is an infinite number of solutions in that case.

5. (BH) Balance the chemical equation
\[
\text{CO}_2 + \text{H}_2\text{O} \rightarrow \text{C}_6\text{H}_{12}\text{O}_6 + \text{O}_2.
\]

Solution. We rewrite the reaction, giving variables to the quantity of each molecule:
\[
x_1\text{CO}_2 + x_2\text{H}_2\text{O} \rightarrow x_3\text{C}_6\text{H}_{12}\text{O}_6 + x_4\text{O}_2.
\]
Balancing the carbon, hydrogen, and oxygen atoms, we have
\[
\begin{align*}
x_1 &= 6x_3 \quad \text{(carbon)} \\
2x_1 + x_2 &= 6x_3 + 2x_4 \quad \text{(oxygen)} \\
2x_2 &= 12x_3 \quad \text{(hydrogen)}
\end{align*}
\]
Therefore, we have that \(x_1 = x_2 = 6x_3\). Substituting these results into the oxygen equation, we have
\[
18x_3 = 6x_3 + 2x_4,
\]
from which we have that \(x_4 = 6x_3\). Note that we have a free variable since any multiple of a balanced reaction will still be balanced. Choosing \(x_3 = 1\) for simplicity, we have
\[
6\text{CO}_2 + 6\text{H}_2\text{O} \rightarrow \text{C}_6\text{H}_{12}\text{O}_6 + 6\text{O}_2.
\]

6. Refer to the figure below.
(a) (MP) Set up and solve a system of linear equations to find the possible flows \(\{f_i\}_i^4\).
(b) (BH) If traffic is regulated on \(CD\) so that \(f_4 = 10\) vehicles per minute, what will the average flows on the other streets be?

Solution. The flow on \(CD\) is \(f_4\). By our answer to part (a), we have that
\[
f_3 = 30 - f_4 = 20, \quad f_2 = 25 - f_4 = 15, \quad f_1 = f_4 - 5 = 5.
\]
(c) (BH) What are the minimum and maximum flows on each street?

Solution. Since the streets are one-way, that means that each of the $f_j$ must be positive. This forces $f_4 \geq 5$ (from the third equation in (B)) and $f_4 \leq 25$ (from the second equation). Then using this result in (B), we have

$$5 \leq f_3 \leq 25, \quad 0 \leq f_2 \leq 20, \quad 0 \leq f_1 \leq 20.$$

(d) (BH) How would the solution change if all of the directions were reversed?

Solution. If all directions are reversed, then the roles of inflow and outflow are reversed. But that just exchanges the left- and right-hand sides of our system, which doesn’t affect the solution. So the solution is the same.

7. (BH) You are given three alloys of the following composition:
Alloy A: 5 parts (by weight) gold, 2 silver, 1 lead
Alloy B: 2 parts gold, 5 silver, 1 lead
Alloy C: 3 parts gold, 1 silver, 4 lead

(a) How much of each metal is in one ounce of Alloy A?

Solution. By the problem statement, for every eight parts of alloy A, there are 5 parts of gold, 2 of silver, and 1 of lead. Therefore, in every ounce of alloy A, there must be $5/8$ ounces of gold, $1/4$ ounce of silver, and $1/8$ ounce of lead.

Suppose we want to make 27 ounces of a new alloy containing equal quantities (by weight) of gold, silver, and lead.

(b) Write the system of equations needed to solve for the amounts of A, B, and C necessary to make this new alloy.

Solution. In this new alloy, we must have nine ounces of each metal. Using the same argument as in part (a), we see that for every ounce of alloy B, there must be $1/4$ ounce
of gold, 5/8 ounces of silver, and 1/8 ounce of lead, and for every ounce of alloy $C$, there must be 3/8 ounces of gold, 1/8 ounce of silver, and 1/2 ounce of lead. We balance the amounts of each element, obtaining

\[
\begin{align*}
\frac{5A}{8} + \frac{B}{4} + \frac{3C}{8} &= 9, \\
\frac{A}{4} + \frac{5B}{8} + \frac{C}{8} &= 9, \\
\frac{A}{8} + \frac{B}{8} + \frac{C}{2} &= 9.
\end{align*}
\]

(c) Solve the system.

**Solution.** As a first step, we multiply each equation by 8 to get rid of the fractions:

\[
\begin{align*}
\begin{bmatrix} a & 5 & 2 & 3 & 72 \\ b & 2 & 5 & 1 & 72 \\ c & 1 & 1 & 4 & 72 \end{bmatrix} \sim \begin{bmatrix} d = a - 2c & 3 & 0 & -5 & -72 \\ e = a - 5c & 0 & -3 & -17 & -288 \\ f = b - 2c & 0 & 3 & -7 & -72 \end{bmatrix}
\end{align*}
\]

Solving the last line, we have that $C = 15$. Back solving, we have $B + 85 = 96$, so $B = 11$, and $A - 25 = -24$, so $A = 1$.

8. Suppose that the following facts are true:

1. Of the number of people $x_s$ who start a year living on the East Coast, 80% stay in and 20% move out during the course of that year.
2. Of the number of people $y_s$ who start a year living off the East Coast, 90% stay out and 10% move in during the course of that year.
3. The number of births and deaths cancel one another out (so you don’t have to worry about them).

Let $x_e$ and $y_e$ be the number of people living on and off the East Coast, respectively, at the end of the year.

(a) (BH) Write down equations needed to solve for $x_e$ and $y_e$ as functions of $x_s$ and $y_s$.

**Solution.** We balance the number of people who end up in each place:

\[
\begin{align*}
0.8x_s + 0.1y_s &= x_e, \\
0.2x_s + 0.9y_s &= y_e.
\end{align*}
\]

The system in (a) is an example of a *Markov process*.

(b) (MP) If $y_e = 100$ million and $x_e = 75$ million, find $x_s$ and $y_s$. 
(c) (BH) If \( x_s = x_e \) and \( y_s = y_e \), find the ratio \( y_e / x_e \).

**Solution.** Substituting the given equalities into (C), we have
\[
\begin{align*}
0.8x_s + 0.1y_s &= x_s \\
0.2x_s + 0.9y_s &= y_s
\end{align*}
\]
\[
\Rightarrow
\begin{align*}
-0.2x_s + 0.1y_s &= 0 \\
0.2x_s - 0.1y_s &= 0
\end{align*}
\]
\[
\Rightarrow
\frac{y_s}{x_s} = \frac{0.2}{0.1} = 2,
\]
where in the last step we have used the fact that the equations are redundant.

In the case given by (c), \((x_s, y_s)\) is called an *eigenvector* of the linear system. We will study this concept in great detail later.

9. (BH) Let \( A \in \mathbb{R}^{m \times n} \), \( x, y \in \mathbb{R}^n \). Using the definition of matrix-vector multiplication in terms of the entries, show that \( A(2x - y) = 2Ax - Ay \).

**Solution.** From notes in class, the \( i \)th entry of \( A(2x - y) \) is given by
\[
\sum_{j=1}^{n} a_{ij}(2x_j - y_j) = 2 \sum_{j=1}^{n} a_{ij}x_j - \sum_{j=1}^{n} a_{ij}y_j.
\]
But the first sum is the \( i \)th entry of \( 2Ax \), and the second sum is the \( i \)th entry of \( Ay \), so this is also the \( i \)th entry of \( 2Ax - Ay \), as required.

10. (BH) Let \( A \in \mathbb{R}^{n \times n} \), \( z \in \mathbb{R}^n \). Consider the equation \( Az = \lambda z \).

(a) Using the definition of matrix-vector multiplication in terms of the entries, show that this equation is equivalent to \( (A - \lambda I)z = 0 \).

**Solution.** It is most convenient to work backwards. First we note that the \( ij \)th entry of \( I \) is given by
\[
\delta_{ij} = \begin{cases} 
1, & i = j, \\
0, & i \neq j.
\end{cases}
\]
(This is sometimes called the *Kronecker delta function.*) Hence by notes in class, the \( i \)th entry of \( (A - \lambda I)z = 0 \) is given by
\[
\sum_{j=1}^{n} (a_{ij} - \lambda \delta_{ij})z_j = 0
\]
\[
\sum_{j=1}^{n} a_{ij}z_j = \lambda \sum_{j=1}^{n} \delta_{ij}z_j.
\]
But when we do the sum on the right-hand side, the only term that survives is \( z_i \), so we have
\[
\sum_{j=1}^{n} a_{ij}z_j = \lambda z_i.
\]
But the left-hand side is the \( i \)th entry of \( Az \), and the right-hand side is the \( i \)th entry of \( \lambda z \). Hence \( Az = \lambda z \), as required.
Now consider the special case where

\[ A = \begin{pmatrix} 4 & -3 \\ 2 & -3 \end{pmatrix}. \]

(b) Show that in order for \( z \neq 0 \), \( \lambda \) must be one of two special values, and calculate those values.

\textit{Solution.}

\[ (A - \lambda I)z = 0 \]

\[ \begin{pmatrix} 4 - \lambda & -3 \\ 2 & -3 - \lambda \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \]

\[ (4 - \lambda)z_1 - 3z_2 = 0 \]
\[ 2z_1 - (3 + \lambda)z_2 = 0 \]

Since the right-hand side is 0, the only way to get nonzero answers is if

\[ (4 - \lambda)(3 + \lambda) - 6 = 0 \]
\[ 12 + \lambda - \lambda^2 - 6 = 0 \]
\[ \lambda^2 - \lambda - 6 = (\lambda + 3)(\lambda - 2) = 0, \]

so \( \lambda \) must be \( -3 \) or 2.

These special values of \( \lambda \) are called \textit{eigenvalues}, and they will become very important later on.
In[1]:= Quit[]

In[1]:= $PrePrint = If[MatrixQ[#] || VectorQ[#], MatrixForm[#], #] &;

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HW1 (Checked)

HW2 (Checked)

HW3 (Checked)

Check

Number 4.

Solve the linear system associated with the following matrices and vectors (here we break up the augmented matrix):

In[2]:= mat4a = {{1, 2, 3, 1}, {1, 3, 0, 1}, {1, 0, 2, 1}}
vec4a = {8, 7, 3}

Out[2]=
\[
\begin{pmatrix}
1 & 2 & 3 & 1 \\
1 & 3 & 0 & 1 \\
1 & 0 & 2 & 1 \\
\end{pmatrix}
\]

Out[3]=
\[
\begin{pmatrix}
8 \\
7 \\
3 \\
\end{pmatrix}
\]

Note that in order to get the most complete solution possible, we use the Solve command:

In[4]:= vars = {x1, x2, x3, x4}
Solve[mat4a.vars == vec4a, vars]

Out[4]=
\[
\begin{pmatrix}
x1 \\
x2 \\
x3 \\
x4 \\
\end{pmatrix}
\]

*** Solve: Equations may not give solutions for all "solve" variables.

Out[5]= ( x2 → 2 x3 → 1 x4 → 1 - x1 )
In[6]:= 
mat4b = {{1, 1, 3, -3}, {0, 2, 1, 3}, {1, 0, 2, -1}}  
vec4b = {0, 3, -1}  
vars = {x1, x2, x3, x4}  
Solve[mat4b.vars = vec4b, vars]

Out[6]=

Out[7]=

Out[8]=

... Solve: Equations may not give solutions for all "solve" variables.

Out[9]= (x2 \rightarrow \frac{21}{13} + \frac{5}{13} x1, x3 \rightarrow \frac{6}{13} - \frac{7}{13} x1, x4 \rightarrow \frac{1}{13} - \frac{x1}{13})

Number 7a.

First we balance the inflow and outflow at each of the nodes. The node balanced is referenced in the equation label.

In[10]=
eq7a = 10 + 10 == f1 + f2  
eq7b = f1 + f3 == 20 + 5  
eq7c = f2 + f4 == 10 + 15  
eq7d = 15 + 15 == f3 + f4

Out[10]= 20 = f1 + f2


Out[12]= f2 + f4 = 25

Out[13]= 30 = f3 + f4

Then we solve the problem.

In[14]= Solve[{eq7a, eq7b, eq7c, eq7d}, {f1, f2, f3, f4}]

... Solve: Equations may not give solutions for all "solve" variables.

Out[14]= (f2 \rightarrow - f1, f3 \rightarrow 25 - f1, f4 \rightarrow 5 + f1)

Note that we have a free variable.

Number 9b.

First we enter the answer from 9(a).
In[15]:= eq9 = {0.8 \* xs + 0.1 \* ys == xe, 0.2 \* xs + 0.9 \* ys == ye}

Out[15]= 

\[
\begin{align*}
0.8 \times xs + 0.1 \times ys &= xe \\
0.2 \times xs + 0.9 \times ys &= ye
\end{align*}
\]

Then we substitute in the values given and solve.

In[16]:= eq9 /. {xe -> 75000000, ye -> 100000000}
Solve[%, {xs, ys}]

Out[16]= 

\[
\begin{align*}
0.8 \times xs + 0.1 \times ys &= 75000000 \\
0.2 \times xs + 0.9 \times ys &= 100000000
\end{align*}
\]

Out[17]= 

\[
\begin{align*}
x_\text{s} \rightarrow 8.21429 \times 10^7 & \\
y_\text{s} \rightarrow 9.28571 \times 10^7
\end{align*}
\]

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**HW4** (Checked)

**HW5** (Checked)

**HW6** (Checked)

**HW7** (Checked)

**HW8** (Checked)

**HW9** (Checked)

**SSM** (Checked)