Homework Set 2 Solutions

1. (a) (BH) Find \( c \) so that the vectors
\[
\begin{pmatrix} c \\ 4 \\ \end{pmatrix} \text{ and } \begin{pmatrix} 2 \\ 5 \end{pmatrix}
\]
are parallel.

Solution. Two vectors are parallel if they are multiples of one another, so we solve

\[
\begin{pmatrix} c \\ 4 \end{pmatrix} = s \begin{pmatrix} 2 \\ 5 \end{pmatrix} \implies s = \frac{4}{5}, c = \frac{8}{5}.
\]

(b) (MP) Calculate the angle between the two vectors for any \( c \).

2. Show that for any nonzero constant \( \alpha \), \( \text{proj}_w v = \text{proj}_{\alpha w} v \).

Solution. We know that

\[
\text{proj}_{\alpha w} v = \frac{v \cdot (\alpha w)}{||\alpha w||^2} (\alpha w) = \frac{\alpha^2}{||w||^2} v \cdot w w = \frac{v \cdot w}{||w||^2} w = \text{proj}_w v.
\]

3. (BH) Let
\[
v = \begin{pmatrix} 5 \\ 4 \end{pmatrix}, \quad w = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.
\]

Solution. We know that

\[
\text{proj}_w v = \frac{v \cdot w}{||w||^2} w = \frac{5 - 4}{2} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1/2 \\ -1/2 \end{pmatrix}.
\]

(b) Find the vector component of \( v \) orthogonal to \( w \).

Solution. The vector component of \( v \) orthogonal to \( w \) (call it \( z \)) is what is left when you subtract off the projection, so we have

\[
z = v - \text{proj}_w v = \begin{pmatrix} 5 \\ 4 \end{pmatrix} - \begin{pmatrix} 1/2 \\ -1/2 \end{pmatrix} = \begin{pmatrix} 9/2 \\ 9/2 \end{pmatrix}.
\]

4. (MP) Let \( v = (-1, 3, 2, 0, 6), w = (5, 3, 2, 1, 0) \).

(a) Find \( \text{proj}_w v \).

(b) Find the vector component of \( v \) orthogonal to \( w \).
5. (BH) Consider the figure at right. Here $||\mathbf{w}_1|| = ||\mathbf{w}_2||$.

(a) Show that $\mathbf{v} \cdot \mathbf{w}_2 = -\mathbf{v} \cdot \mathbf{w}_1$.

*Solution.* By the figure, we have that $\mathbf{w}_2 = \mathbf{v} + \mathbf{w}_1$. Therefore

$$\mathbf{v} \cdot \mathbf{w}_1 = (\mathbf{w}_2 - \mathbf{w}_1) \cdot \mathbf{w}_1 = \mathbf{w}_2 \cdot \mathbf{w}_1 - ||\mathbf{w}_1||^2,$$

$$\mathbf{v} \cdot \mathbf{w}_2 = (\mathbf{w}_2 - \mathbf{w}_1) \cdot \mathbf{w}_2 = ||\mathbf{w}_2||^2 - \mathbf{w}_2 \cdot \mathbf{w}_1.$$  

Since the lengths of the vectors are equal, the result is proved.

(b) Prove that the angles at the base of an isosceles triangle are equal.

*Solution.* We know that

$$\cos \theta_2 = \frac{\mathbf{v} \cdot \mathbf{w}_2}{||\mathbf{v}|| ||\mathbf{w}_2||}, \quad \cos \theta_1 = \frac{(-\mathbf{v}) \cdot \mathbf{w}_2}{||-\mathbf{v}|| ||\mathbf{w}_2||}.$$  

By part (a), we have that the numerators are equal, and the denominators are equal because the lengths of the sides are equal. Hence $\cos \theta_2 = \cos \theta_1$, and hence the angles are equal.

6. (BH) Consider the following system:

$$x - 2y = 7, \quad \text{(A.1)}$$

$$3x + y = 7. \quad \text{(A.2)}$$

(a) Determine geometrically whether the system has a unique solution, infinitely many solutions, or no solution.

*Solution.* See figure below. The system has a unique solution.
(b) Solve the system algebraically.

**Solution.** (There are many correct ways to do this problem.) If we add twice (A.2) to (A.1), we eliminate \( y \) and obtain

\[
7x = 21 \quad \Rightarrow \quad x = 3.
\]

Substituting this result into either equation, we obtain \( y = -2 \).

7. (MP) Consider the following system:

\[
0.10x - 0.05y = 0.20, \\
-0.06x + 0.03y = -0.12.
\]

(a) Determine geometrically whether the system has a unique solution, infinitely many solutions, or no solution.
(b) Solve the system algebraically.

8. (BH)

(a) Find a system of two linear equations in the variables \( x_1, x_2, x_3 \) whose solution set is given by the parametric equations

\[
x_1 = t, \quad x_2 = 1 + t, \quad x_3 = 2 - t.
\]

**Solution.** (There are many correct ways to do this problem.) We need to construct two equations which are not redundant. The easiest way to do this is to omit one variable in each. Taking \( x_1 \) and \( x_3 \), we have

\[
x_1 + x_3 = 2.
\]

Taking \( x_2 \) and \( x_3 \), we have

\[
x_2 + x_3 = 3.
\]

(b) Find another parametric solution to the system in part (a) where the parameter is \( s \) and \( x_3 = s \).

**Solution.** Setting \( x_3 = s \) is equivalent to setting \( s = 2 - t \), or \( t = 2 - s \). Then we have

\[
x_1 = t = 2 - s, \quad x_2 = 1 + t = 3 - s.
\]

9. (BH) Consider the following system:

\[
x - 3y = \lambda, \quad (2.1a) \\
-2x + 6y = 5. \quad (2.1b)
\]

For which value(s) of \( \lambda \) will (2.1) have
(a) more than one solution?
(b) exactly one solution?
(c) no solution?

Solution. (There are many correct ways to do this problem.) If we add twice (2.1a) to (2.1b), we will eliminate both $x$ and $y$ from the resulting equations:

$$0 = 2\lambda + 5.$$ 

Therefore, the problem has no solution if $\lambda \neq -5/2$. If $\lambda = -5/2$, we have

$$x - 3y = -5/2$$
$$-2x + 6y = 5,$$

which are redundant equations. Therefore, we really have one equation in two unknowns, which results in an infinite number of solutions. The problem never has exactly one solution.

10. (BH) Solve the following system by the method of elimination:

$$2x_1 + 3x_2 - x_3 = 6,$$  

(2.2a)

$$2x_1 - x_2 + 2x_3 = -8,$$  

(2.2b)

$$3x_1 - x_2 + x_3 = -7.$$  

(2.2c)

Solution. (There are many correct ways to do this problem.) If we add (a) to 3 times each of (b) and (c), we will eliminate $x_2$ from the resulting equations:

$$8x_1 + 5x_3 = -18,$$  

(d)

$$11x_1 + 2x_3 = -15.$$  

(e)

Then taking 2(d) − 5(e), we obtain

$$-39x_1 = 39 \quad \implies \quad x_1 = -1,$$

$$2x_3 = -4 \quad \implies \quad x_3 = -2,$$

where we have substituted $x_1$ into (e) to obtain $x_3$. Then substituting both results into (c), we have

$$-x_2 - 5 = -7 \quad \implies \quad x_2 = 2.$$
In[1]:= Quit[]

In[1]:= $PrePrint = If[MatrixQ[#] || VectorQ[#, MatrixForm[#]], #] &;

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HW1 (Checked)

HW2 (Checked)

Check

Number 1b.
Find the angle between the following vectors:

\[
\text{In}[2]:= \text{v1} = \{c, 4\} \\
\text{v2} = \{2, 5\} \\
\text{VectorAngle}[\text{v1, v2}] \\
\text{Out}[2]= \left(\begin{array}{c}
  c \\
  4
\end{array}\right) \\
\text{Out}[3]= \left(\begin{array}{c}
  2 \\
  5
\end{array}\right) \\
\text{Out}[4]= \cos^{-1}\left(\frac{2c + 20}{\sqrt{29}\sqrt{c^2 + 16}}\right)
\]

Number 4.
Let \(v = (-1, 3, 2, 0, 6)\) and \(w = (5, 3, 2, 1, 0)\). Find the vector projection of \(v\) onto \(w\) and the vector component of \(v\) orthogonal to \(w\).
Consider 7.

Consider the system defined below:

\[
\begin{align*}
\text{In}[9] &= \text{sys7} = \{0.10 \times x - 0.05 \times y &= 0.2, -0.06 \times x + 0.03 \times y &= -0.12\}\\
\text{Out}[9] &= \begin{cases} 0.1 \times x - 0.05 \times y &= 0.2 \\ 0.03 \times y - 0.06 \times x &= -0.12 \end{cases}
\end{align*}
\]

7a. Determine geometrically whether the system has a unique solution, infinitely many solutions, or no solution.
\[\text{In[10]} = \{\text{Solve}[\text{sys7}[[1]], y], \text{Solve}[\text{sys7}[[2]], y]\}\]

\[\text{Solve}: \text{Equations may not give solutions for all "solve" variables.}\]

\[\text{Out[10]} = \{y \rightarrow -20. (0.2 - 0.1 x)] \]

\[\{y \rightarrow 33.3333 (0.06 x - 0.12)\}\]

The lines lie on top of one another, so we should expect infinitely many solutions.

7b. Solve the system algebraically.

\[\text{In[12]} = \text{Solve}[\text{sys7}, \{x, y\}]\]

\[\text{Out[12]} = \{y \rightarrow 2. x - 4.\}\]

Note that the answer has a free variable, so there are infinitely many solutions.

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HW3 (Checked)

HW4 (Checked)

HW5 (Checked)

HW6 (Checked)

HW7 (Checked)

HW8 (Checked)

HW9 (Checked)