Homework Set 5

Read sections 3.5, 6.1, 6.2.

Section 3.5/6.1

1. (BH) Let $V$ be the set of all real numbers, and define vector addition by $u \oplus v = 2(u + v)$ (for all $u, v \in V$) and scalar multiplication by $c \odot u = cu$. Check to see if this combination is a vector space. Check all properties, noting which hold and which do not.

2. (BH) In each part, consider the given subset of the vector space $P_2$. Is the subset a subspace?
   
   (a) $\{a_2 t^2 + a_1 t + a_0, \ a_0 = a_1 = 0\}$
   
   (b) $\{a_2 t^2 + a_1 t + a_0, \ a_1 = 2a_0\}$
   
   (c) $\{a_2 t^2 + a_1 t + a_0, \ a_2 + a_1 + a_0 = 2\}$

Section 3.5

3. (BH) Let $A \in \mathbb{R}^{3 \times 5}$.
   
   (a) Give all possible values for rank $A$.
   
   (b) If rank $A = 3$, what is the dimension of its column space?
   
   (c) If rank $A = 3$, what is the dimension of the solution space of the homogeneous system $Ax = 0$?

4. (BH) Let

   \[
   A = \begin{pmatrix}
   1 & 2 & 3 & 2 & 1 \\
   0 & 5 & 4 & 0 & -1 \\
   2 & -1 & 2 & 4 & 3
   \end{pmatrix}.
   \]

   (a) Write a basis for $N(A)$, the null space of $A$.
   
   (b) Write a basis for $\text{col} \ A$.
   
   (c) Write a basis for $\text{row} \ A$. Verify that $\dim \text{row} \ A = \dim \text{col} \ A$.
   
   (d) Verify that $\dim N(A) + \text{rank} \ A = n$, the number of columns of the matrix.

5. (MP) Repeat the steps in #4 for the matrix

   \[
   A = \begin{pmatrix}
   1 & 1 & -1 & 2 & 0 \\
   2 & -4 & 0 & 1 & 1 \\
   5 & -1 & -3 & 7 & 1 \\
   3 & -9 & 1 & 0 & 2
   \end{pmatrix}.
   \]
6. (BH) Let

\[
B = \begin{pmatrix}
2 & -11 & 8 \\
6 & -3 & 4 \\
8 & -2 & 4
\end{pmatrix}, \quad v = \begin{pmatrix}
-2 \\
4 \\
6
\end{pmatrix}.
\]

(a) Is \(v \in \text{col } B\)?
(b) Is \(v \in \text{N}(B)\)?

7. (BH) Consider the following two sets of equations:

\[
\begin{align*}
2x + 3y + z &= 3 \\
-x + y + 2z &= 6 \\
2x - 4y - 2z &= -2
\end{align*}
\]
\[
\begin{align*}
2x + 3y + z &= -6 \\
-x + y + 2z &= -12 \\
2x - 4y - 2z &= 4
\end{align*}
\]

The system on the left has a solution (you can take this as a given). Explain why the system on the right must have a solution. Do NOT calculate either solution; just explain why it must be so given the theory you have learned in class.

Section 6.2

8. (BH) Let

\[
S = \left\{ \begin{pmatrix}
1 \\
0 \\
1
\end{pmatrix}, \begin{pmatrix}
0 \\
1 \\
0
\end{pmatrix}, \begin{pmatrix}
1 \\
1 \\
1
\end{pmatrix}, \begin{pmatrix}
-1 \\
1 \\
-1
\end{pmatrix} \right\}.
\]

Find a basis for the subspace \(W\) of \(M_{22}\) given by \(W = \text{Span } S\).

9. (BH) Determine whether each of the following sets is linearly independent in the appropriate vector space:

(a) \[\left\{ \begin{pmatrix}
1 \\
2 \\
0
\end{pmatrix}, \begin{pmatrix}
2 \\
1 \\
2
\end{pmatrix}, \begin{pmatrix}
-7 \\
1 \\
-10
\end{pmatrix} \right\}\]
(b) \[\left\{ \begin{pmatrix}
1 \\
2 \\
0
\end{pmatrix}, \begin{pmatrix}
1 \\
5 \\
2
\end{pmatrix} \right\}\]
(c) \{\cos^2 x, \cos 2x, 1\}
(d) \{x^2 + 2x, 2x - 4, 1\}

10. (BH) Which of the following sets of vectors are bases for \(P_2\)?

(a) \{-t^2 + t + 2, 2t^2 + 2t + 3, 4t^2 - 1\}.
(b) \{t^2 + 2t - 1, 2t^2 + 3t - 2\}.
(c) \{t^2 + 1, 3t^2 + 2t + 1, 6t^2 + 6t + 3\}.
(d) \{3t^2 + 2t + 1, t^2 + t + 1, t^2 + 1\}.